# Rotations 2

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## 1 Introduction

There is something special about momentum and energy-their conservation laws-which simply states that if there is nothing to change it, it will not change. The thing that causes momentum and energy to change is the force.

Likewise, there are also similar quantities as listed in the previous chapter for the rotation aspects. This chapter shall focus on solving problems using the concepts explained in the Rotations 1.

## 2 Rotational Statics and Dynamics

When solving problems on forces, what we usually have to do is to equate the forces on one axis and the product of mass and acceleration on that axis. Likewise, for rotational aspects, we just have to equate the torque and the change in angular momentum.

$$\sum \tau = \sum Fr = I\alpha$$

### 2.1 Rolling without Slipping

This is just a special case of motion. Just imagine the scenario where the ball on the floor, merely translating but not rotating. In this case, the ball in just rubbing across the floor and kinetic friction comes into play, leading to energy loss.

In the case of rolling without slipping, the point of contact is instantaneously at rest. While there is friction(static), there is no energy loss. It is similar to the process of walking. Your feet are not moving with respect to the ground, but there is friction to push you forward. The formula for rolling without slipping is as follows:

$$v_{CM} = r\omega$$

where $v_{CM}$ is the velocity of the centre of mass.

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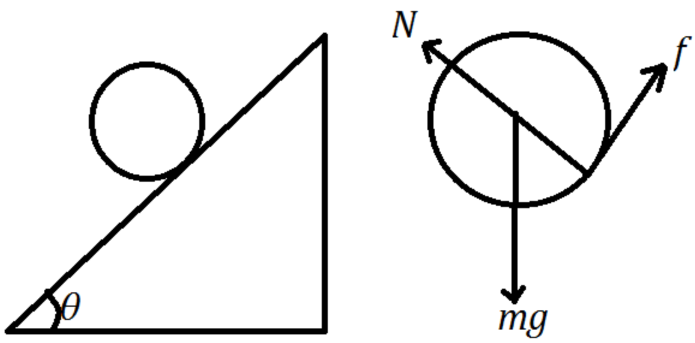
**R1.(USAPhO 2013)**A solid round object of radius $R$ can roll down an incline that makes an angle $\theta$ with the horizontal. Assume that the rotational inertia about an axis through the center of mass is given by $I = \beta m R^2$. The coefficient of kinetic and static friction between the object and the incline is $\mu$. The object moves from rest through a vertical distance $h$.

(a) If the angle of the incline is suffciently large, then the object will slip and roll; if the angle of the incline is suffciently small, then the object with roll without slipping. Determine the angle $\theta_c$ that separates the two types of motion.

(b) Derive expressions for the linear acceleration of the object down the ramp in the case of i. rolling without slipping, and ii. rolling and slipping.

Without saying anything, start by drawing a free body diagram.




We shall resolve the weight into components parallel(like friction $f$) and perpendicular(like the normal force) to the incline. We know that $N = mg\ cos\theta$, or else it means the object has a net acceleration perpendicular to the plane, so the object will fly out of the plane or sink into it.

The net linear acceleration parallel to the plane is such that

$$ma = mg\ sin\theta - f$$

The net angular acceleration about the centre of mass is such that

$$\beta m R^2 \alpha = \tau = fR$$

For rolling without slipping to occur, $a = R\alpha$. This is necessary for the $v_{CM}$ to remain equal to $R\omega$. When rolling without slipping, the static friction is such that $f \leq \mu mg\ cos\theta$. Combining the equations we got:

$$ma = mg\ sin\theta - \beta m R\alpha = mg\ sin\theta - \beta ma \Longrightarrow a = \frac{g\ sin\theta}{1+\beta}$$

$$f = \beta m R\alpha = \beta ma = \frac{\beta mg\ sin\theta}{1+\beta} \leq \mu mg\ cos\theta$$

$$tan\theta \leq \mu(1+\frac{1}{\beta})$$

$\theta_C$ is when equality holds. Beyond this value, friction turns into kinetic friction and the friction is no longer sufficient for it to rotate fast enough for rolling without slipping. This gives us the answer to (a)$\theta_C = tan^{-1}(\mu(1+\frac{1}{\beta}))$.

We have already gotten the answer for rolling without slipping for an angle $\theta \leq \theta_C$. Answer for (b)i is just $a = \frac{g\,sin\theta}{1+\beta}$. What about $\theta > \theta_C$. Fortunately, it we have knowledge on the value for kinetic friction, that is $f = \mu mg\,cos\theta$.

$$ma = mg\,sin\theta - \mu mg\,cos\theta \implies a = g(sin\theta - \mu g cos\theta)$$

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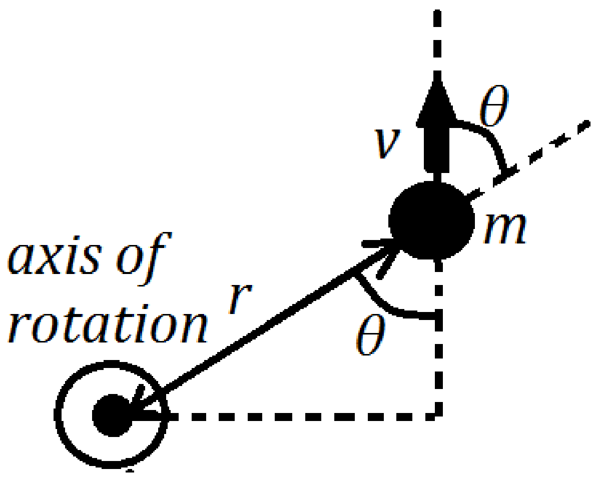
# 3 Conservation of Angular Momentum

Without external influence from forces(or torque), the angular momentum, $L$, is expected to be conserved, as explained in the previous chapter.




$$L = mvr\,sin\theta = rp\,sin\theta$$

Or if we were to take the case where a rigid body is rotating about a point, $\theta = 90°$

$$L = mvr = mr^2\omega = I\omega$$

where $I = mr^2$ is the moment of inertia. To know whether angular momentum is conserved, we just have to ask whether there is an external torque(whether there is an external that **can cause a change in rotational motion**).

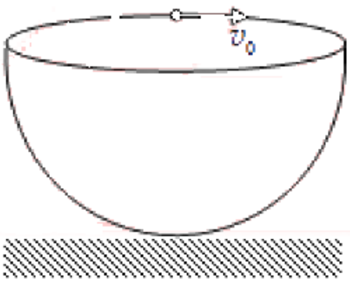
(This source explains better than me: Youtube minutephysics "What IS Angular Momentum?")

A very common example used to explain this concept is the case where a spinning ice skater pulls in his or her hands to spin faster. Another example is the Hoberman sphere as illustrated in this video(Youtube Physics Demos "Angular Momentum Demo: Hoberman Sphere").

In both examples, the external torque(probably due to air resistance) is negligible. When the skater pulls in his or her hands or when the ball in the contracted state, the overall value for $r$ decreases, which leads to a decrease in moment of inertia $I$. From the conservation of momentum, we can know that $\omega$ will increase, causing the corresponding objects to spin faster.

In general, the steps involve finding an axis of rotation about which there is no external cause of torque, and then equate the initial and final angular momentum about this axis.

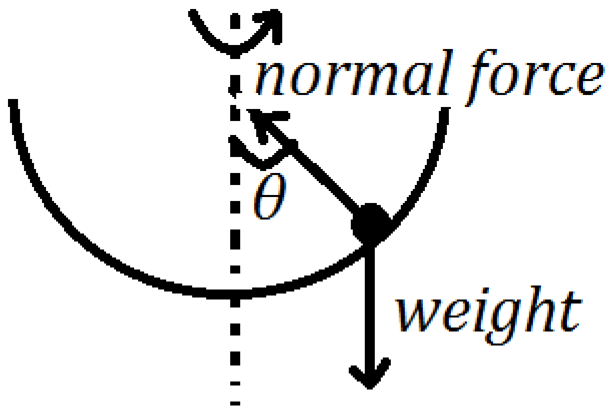
**R2.** A hemispherical bowl of radius $R$ with a frictionless inner surface is fixed to the ground. At the rim of the bowl, a point particle is given a horizontal velocity of $v_0$. Find the maximum velocity which the particle can attain. The acceleration of gravity is $g$.




Let's understand the system. The particle travels circularly downwards under the influence of gravity. As the bowl is frictionless, no energy is loss and we can use the principle of conservation of energy. Suppose the bottom of the bowl has height zero and the height at any instant is $h$, conservation of energy gives us

$$\frac{1}{2}mv_0^2 + mgR = \frac{1}{2}mv_f^2 + mgh = \frac{1}{2}mv_f^2 + mgR(1 - cos\theta)$$

This gives us one equation with two unknowns $v$ and $\theta$, which is defined in the next diagram. We need one more equation and given the ball is moving in a circle, we can probably guess the equation comes from the conservation of angular momentum. But first we need to see whether there is any torque, which requires us to look at the forces on the particle.




As the particle moves down, it is not just rotating about the vertical centre axis but also about some horizontal axis. But nonetheless, if we were to only regard the component of angular momentum corresponding to the vertical axis, it should still be conserved. In order for the particle to rotate faster about the vertical axis, there needs to be a force in the horizontal plane(Imagine this in your mind, if an object rotates about the $xy$ axis, the angular momentum vector is in the $z$ axis, this can be known from $\vec{r} \times \vec{p}$. Refer to the notes on vectors.) However, from the force diagram, there is no such force, which leads to conservation of angular momentum about the vertical axis.

$$mRv_0 = mR\,sin\theta\,v_f$$

The rest is just Math. Solving the 2 equations simultaneously

$$\frac{1}{2}sin^2\theta\,v_f^2 + gR = \frac{1}{2}v_f^2 + gR(1 - cos\theta)$$

$$gRcos\theta = \frac{1}{2}cos^2\theta\,v_f^2$$

$$v_f^2 = \frac{2gR}{cos\theta} = \frac{2gR}{\sqrt{1 - (\frac{v_0}{v_f})^2}} \Longrightarrow v_f^4 - v_0^2 v_f^2 - 4g^2R^2 = 0$$

Solving the quadratic equation and eliminating the negative solution.

$$v_f^2 = \frac{v_0^2 + \sqrt{v_0^4 + 16g^2R^2}}{2} \Longrightarrow v_f = \sqrt{\frac{1}{2}(v_0^2 + \sqrt{v_0^4 + 16g^2R^2})}$$

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# 4 Rotational Kinetic Energy

Rotation is a form of movement and also requires(or implies the presence of) kinetic energy. When referring to rotational kinetic energy, we are really referring to the sum of the kinetic energy of every single particle of the object. Let's first consider an object rolling about its centre of mass at angular velocity $\omega$.

$$KE = \sum \frac{1}{2}mv^2 = \sum \frac{1}{2}mr^2\omega^2 = \frac{1}{2}\omega^2 \sum mr^2 = \frac{1}{2}I\omega^2$$

This formula looks similar to the formula for linear kinetic energy.

What if an object is both translating and rotating, how will its total kinetic energy by calculated? Consider an already rotating object inside a massless, frictionless box. It has a kinetic energy $KE_{rotational}$ already. Then, I exert a force on the box to cause the whole object to move. There would have been a gain in kinetic energy.

If you find it odd to add this "box", you can just exert a force towards the CM. In this way, it will not affect the rotational motion, but will affect the translational motion.

From this we can deduce that the total kinetic energy is the sum of rotational kinetic energy about the centre of mass and the kinetic energy due to the translation of the centre of mass.

$$KE_{total} = KE_{\text{rotation about CM}} + KE_{\text{translation of CM}}$$

# 5 Rotation about CM + Translation of CM

Most things in rotation require breaking down the system into rotation about the CM and translation of the CM. I have shown this in energy, but something similar can also be done for angular momentum. I have left this to the last as I feel it is quite complicated and I probably will not be able to understand at Secondary 3 or 4.

I'll first write down the result and leave the derivation to the back as the derivation I know of requires quite a bit of vectors.

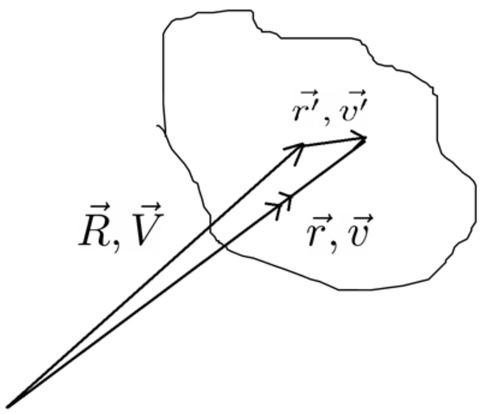
Consider an object that is translating about an external point and at the same time also rotating about its centre of mass. The total angular momentum is given by the sum of the angular momentum of the object about this external point as if all mass is concentrated at its centre of mass, and the angular momentum of the object about its centre of mass.

$$L_{total} = L_{\text{translation of CM}} + L_{\text{rotation about CM}}$$

Now here's the derivation :) I'll take all position vectors to be with respect to the origin. I'll define a few variables. $\vec{V}, \vec{R}$ are the velocity and position vector of the CM. $\vec{v'}, \vec{r'}$ and the velocity and position vector of a point mass of the object with respect to the CM. Therefore, the velocity and position of the point mass with respect to the origin are $\vec{V} + \vec{v'}$ and $\vec{R} + \vec{r'}$ respectively.

Therefore, the total angular momentum is given by the sum of the angular momentum of each individual point mass. :

$$\vec{L} = \sum m\vec{r} \times \vec{v} = \sum m(\vec{R} + \vec{r'}) \times (\vec{V} + \vec{v'})$$




The $\times$ represents cross product in vectors, but it is similar to the normal multiplication.

$$\vec{L} = \sum m \left( \vec{R} \times \vec{V} + \vec{R} \times \vec{v'} + \vec{r'} \times \vec{V} + \vec{r'} \times \vec{v'} \right) = \sum m(\vec{R} \times \vec{V} + \vec{r'} \times \vec{v'})$$

The values of $\sum m\vec{R} \times \vec{v'}$ and $\sum m\vec{r'} \times \vec{V}$ all happen to be zero, based on the definition of the centre of mass, where $\sum m\vec{r'}$ and $\sum m\vec{v'}$ all gives zero.

The first term $\sum m\vec{R} \times \vec{V}$ gives the angular momentum of the centre of mass, while $\sum m\vec{r'} \times \vec{v'}$ gives the angular momentum about the centre of mass.