

# Oscillations

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## 1 Introduction

”Perfectly balanced, as all thing should be.” -Thanos

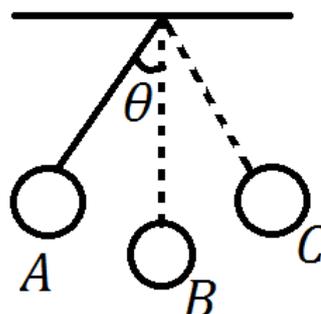
Maybe there is some way for the universe to restore balance. Normally, things won't change, but when something changes, there tends to be something that reduces that change. Consider a pendulum, which when moved from its equilibrium vertical position, will start to swing about the vertical, but never reaching anywhere further from the original height.

By oscillations, I'm referring to situations like this, where something or some value(it may not necessarily be a physical object) moves back and forth about a certain point or value.

## 2 What, Where, When

A lot of systems undergo oscillation, even those that are not physically moving. Anyways, in oscillating systems, there are a few questions that we should ask.

First question that I would ask is: what is oscillating/what is the quantity that we are interested in that is changing periodically? For a pendulum, the value is  $\theta$ , the angular position, the angle which the pendulum makes with respect to the vertical.



Take note that the quantity may differ for different systems. For a spring-and-mass system, the quantity is the distance from the equilibrium position. For electrical circuits, the quantity may be the voltage.

The second question is where, which is relevant to the position(or the value of the quantity). For all oscillations(that I know of), there is an equilibrium position where the object experiences no net force. In the case of the pendulum, it is the bottom of the swinging motion(B). At this point, an object which starts off at rest will remain at rest, while displacement from it will lead to oscillation. Another position is the extreme position(A and C), where the net displacement is the greatest. This maximum value of displacement is the amplitude.

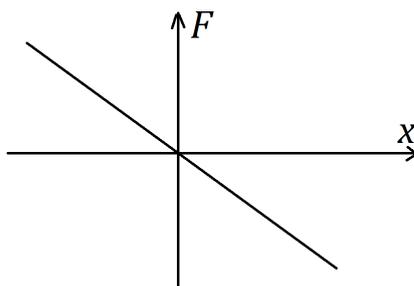
The third question to ask is when. It is important to note not just where the object is because it is not at the same position forever. Its position will change with respect to time. One important time quantity is the period,  $T$ , which is the time taken to complete one whole cycle or oscillation. An oscillation is the period of motion which is being repeated. For the pendulum, the period is the time taken for it to move from A to B to C and back to A again.

There is a quantity related to period, that is frequency,  $f$ , which is defined as the number of oscillations per unit time. It should be quite obvious that  $f = \frac{1}{T}$ .

### 3 Simple Harmonic Motion

This is the most commonly discussed type of oscillation. It is probably also the only type which we need to be familiar with. I believe it is commonly discussed because of two reasons: it is common in the real world/it is a good approximation of the real world, and that it is Mathematically simpler to deal with.

Let's start with some definition. Simple harmonic motion occurs when the restoring force,  $F$ , on an object is **directly proportional** to its displacement,  $x$  from equilibrium, that is



$$F = -kx$$

The negative sign implies that it is a restoring force to "pull" back the object. If  $x$  is positive,  $F$  becomes negative so the object can return to position 0. Vice versa for when  $x$  is negative.

If you realise, the formula for the force is exactly the same as that for a spring, which obeys Hooke's Law. Indeed, a spring-and-mass system is one of the systems that is truly simple harmonic(that is it is not just an approximation).

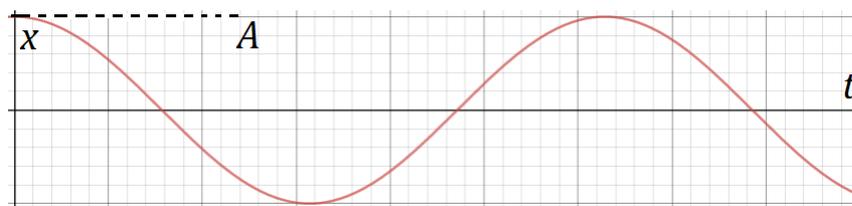
## 4 Some Mathematical Analysis

We shall consider the case of a mass attached to a spring. When displaced by  $x$  from equilibrium, it experiences a force of  $-kx$ . Our aim here is to find  $x$  as a function of time,  $t$ , so we know how the displacement varies with time.

From Newton's 2nd Law, I can rewrite the force equation like this:

$$F = m \frac{d^2x}{dt^2} = m\ddot{x} = -kx$$

If you have done calculus, you may realise that the solutions to this equation can be sine or cosine. Let's just consider the case where  $x = A \cos(\omega t + \phi)$ .  $A$  is the maximum displacement(i.e. the amplitude).  $\phi$  depends on the initial conditions(I may start the timer only sometime after oscillation started).  $\omega$  is some constant which we are trying to find.



$$x = A \cos(\omega t + \phi) \quad \frac{dx}{dt} = -A\omega \sin(\omega t + \phi) \quad \frac{d^2x}{dt^2} = -A\omega^2 \cos(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -\omega^2 x = -\frac{k}{m} x \quad \omega = \sqrt{\frac{k}{m}}$$

From the above equation, we can see that  $\omega$  is just the square root of the constant of proportionality between acceleration and displacement.

$\omega$  is called the angular frequency, and it is not just a random number with no meaning. In sine and cosine graphs,  $2\pi$  represents one cycle/oscillation(the function repeats itself every  $2\pi$ ).  $ft$ -product of frequency and time-represents the number of oscillations that passed. For the sine or cosine function to be

able to "react" to the number of oscillations, I'll have to multiply the number of oscillations by  $2\pi$ .  $2\pi ft = \omega t, \omega = 2\pi f$ . Think of  $\omega$  as just a quantity to represent the number of  $2\pi$  per unit time for sine and cosine functions to operate in the way we want.

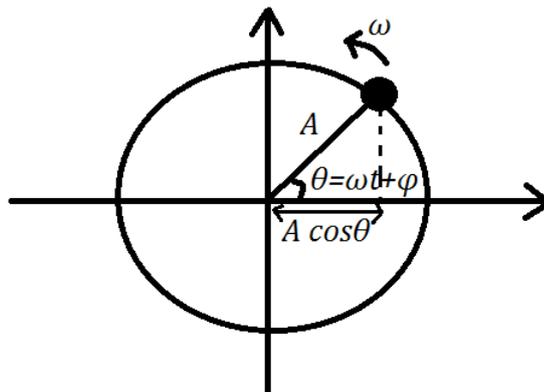
For most problems, the standard way is to find the constant of proportionality between  $\ddot{x}$  and  $x$ , then take a square root to find angular frequency, which can then be used to find the normal frequency. The period of oscillation can then be calculated from  $\frac{1}{f}$  or  $\frac{2\pi}{\omega}$ . For a simple spring-mass system, the period is  $2\pi\sqrt{\frac{m}{k}}$ .

Another perspective to look at is from the energy perspective, which sometimes makes it easier to lay out the equation as compared to using forces. Any system with potential energy similar to that of elastic potential energy (quadratic) is simple harmonic. It can be shown simply where there is no energy loss, by differentiating with respect to time, it leads us to the force equation.

$$\frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 = E \implies kx\dot{x} + m\dot{x}\ddot{x} = 0 \implies m\ddot{x} = -kx$$

## 4.1 A Visualization of $\omega$

An object undergoing simple harmonic motion can be seen as observing an object moving in a circle from the side perspective.



The position,  $x$ , is just the horizontal coordinate of the object rotating about the origin. The rate at which  $\theta$  changes is  $\omega$ .

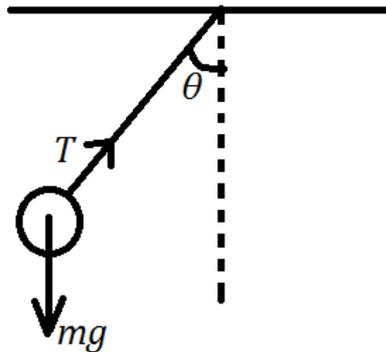
## 4.2 An Approximation

We might have come across the formula for the period of a pendulum as  $2\pi\sqrt{\frac{l}{g}}$  somewhere in our learning, where  $l$  is the length of the string and  $g$  is the acceleration due to gravity. This formula is only approximately true.

Let's analyze the forces on a simple pendulum. Given the string is always taut and inextensible, the pendulum will be moving in a circle. The component of the weight perpendicular to the string is what led to tangential acceleration.

$$mg \sin\theta = ma_{\text{tangential}} = -ml\ddot{\theta}$$

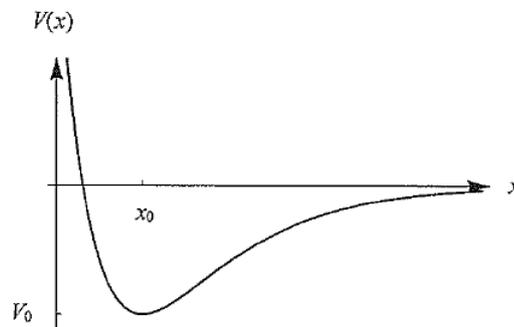
Here comes the approximation part. For  $\theta$  small enough (much smaller than 1),  $\sin\theta \approx \theta$ . This will make more sense after you learn Maclaurin series, but let's just accept this for now. With this, the previous equation can then be approximated as:



$$mg\theta = -ml\ddot{\theta} \quad \ddot{\theta} = -\frac{g}{l}\theta \implies \omega = \sqrt{\frac{g}{l}} \implies T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

Quite a few problems involve performing certain small approximations to make the problem more solvable.

**O1. (SJPO 2018 General Round)** The figure shows the potential energy  $V(x)$  as a function of molecular separation  $x$  for a diatomic molecules of reduced mass  $\mu$ .



If  $V(x) = V_0(1 - e^{-(x-x_0)/\delta})^2 - V_0$ , what is the vibrational frequency at the equilibrium position? (Hint: This series expansion may be useful.  $e^x \approx 1 + x + \frac{x^2}{2!}$ )

I have briefly explained what reduced mass is in gravitation. It has roughly the same idea here: it is the mass I can use to model a two-body system as a

one-body system.

The question said something about the equilibrium position, so it is probably wise to know where it is. To know the force from a potential energy function, we need to find  $F(x) = -\frac{dV}{dx}$ , or the negative potential gradient. At the equilibrium position, there is no net force, which implies a stationary point, which is conveniently labelled as  $x_0$ . It also happens that it is not just any stationary point, it is a minimum point, which is necessary for equilibrium to occur. When  $x < x_0$ , the force pushes it towards  $x_0$ . Vice versa for when  $x > x_0$ .

For convenience, we shall replace  $(x - x_0)$  with  $\Delta x$ , the displacement from equilibrium. This gives us:

$$V(\Delta x) = V_0(1 - e^{-(\Delta x)/\delta})^2 - V_0$$

To find the force, I'll do a differentiation(using Chain rule):

$$F(\Delta x) = -\frac{dV(\Delta x)}{d\Delta x} = -V_0 \cdot 2 \left(1 - e^{-\frac{\Delta x}{\delta}}\right) \cdot \left(\frac{1}{\delta} e^{-\frac{\Delta x}{\delta}}\right) = -\frac{2V_0}{\delta} \left(e^{-\frac{\Delta x}{\delta}} - e^{-\frac{2\Delta x}{\delta}}\right)$$

This looks nothing like what we know about SHM, a sign that we need to do some form of approximation.

$$\begin{aligned} F(\Delta x) &\approx -\frac{2V_0}{\delta} \left[ \left(1 - \frac{\Delta x}{\delta} + \frac{(\Delta x)^2}{2\delta^2}\right) - \left(1 - \frac{2\Delta x}{\delta} + \frac{4(\Delta x)^2}{2\delta^2}\right) \right] \\ F(\Delta x) &\approx -\frac{2V_0}{\delta} \left[ \left(1 - \frac{\Delta x}{\delta} + \frac{(\Delta x)^2}{2\delta^2}\right) - \left(1 - \frac{2\Delta x}{\delta} + \frac{4(\Delta x)^2}{2\delta^2}\right) \right] \\ &= -\frac{2V_0}{\delta} \left( \frac{\Delta x}{\delta} - \frac{3(\Delta x)^2}{2\delta^2} \right) \\ &\approx -\frac{2V_0}{\delta^2} \Delta x \end{aligned} \tag{1}$$

I made the approximation in the last line so that it looks like our SHM equation:  $F(\Delta x) = -k\Delta x$ . If we take the "mass" to be  $\mu$ , our knowledge of the solution of such an equation tells us that the angular frequency can be calculated as  $\sqrt{\frac{k}{\mu}}$ . The vibrational frequency can then be calculated as:

$$f = \frac{1}{2\pi} \sqrt{\frac{2V_0}{\mu\delta^2}}$$

## 5 Damping and Driving

The simple harmonic motion, where there only exists the restoring force, is quite idealistic. There can be other forms of forces.

### 5.1 Damping

Damping occurs when an object experiences a force that opposes it's motion. Therefore, with addition to the restoring force, there is a damping force which is directly proportional to velocity(so not the normal kind of friction).

$$m\ddot{x} = -kx - b\dot{x} \implies \ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x} \implies \ddot{x} = -\omega^2x - \gamma\dot{x}$$

The equation gives different solutions for different value of  $\gamma$  and  $\omega$ , corresponding to different types of damping motion. Let's try to solve this equation(involving calculus and complex numbers, let's just assume there exist a number  $i$  such that  $i^2 = -1$ , you may know of it as the imaginary number). If you want, you can skip this and go to the summary at the end.

Imaginary and real numbers and their sums make up complex numbers, and there's something special about exponents and complex numbers. The famous Euler's formula states that  $e^{i\theta} = \cos\theta + i\sin\theta$ . As it turns out, there is a relation between exponential and the trigonometric functions. I'll assume  $x$  takes on the form  $Ae^{\alpha t}$ , where  $\alpha$  can be a complex number(this step may feel weird but it is quite a common step people use). Substituting this back into the equation for damped motion gives us:

$$\alpha^2 Ae^{\alpha t} = -\omega^2 Ae^{\alpha t} - \gamma\alpha Ae^{\alpha t} \implies \alpha^2 + \gamma\alpha + \omega^2 = 0$$

We can solve for  $\alpha$  using the quadratic formula.

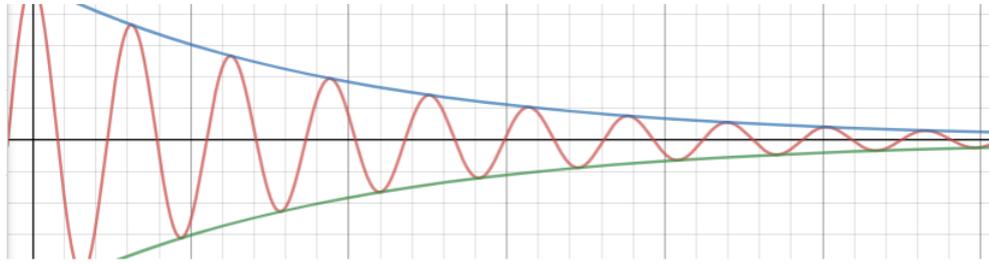
$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega^2}$$

The solution differs depending on whether the discriminant is positive or negative, which decides whether  $\alpha$  is complex or purely real.

**Case 1: when  $\gamma$  is small, the discriminant is negative**, which makes  $\alpha$  complex. There is a real part  $(-\frac{\gamma}{2})$  and an imaginary part  $(\sqrt{\frac{\gamma^2}{4} - \omega^2})$ , making  $x = Ae^{-\frac{\gamma}{2}t}e^{i\sqrt{\omega^2 - \frac{\gamma^2}{4}}t}$ .

$e^{-\frac{\gamma}{2}t}$  creates a decreasing envelope.  $e^{i\sqrt{\omega^2 - \frac{\gamma^2}{4}}t}$  has both a real and imaginary part, but let's just take its real part, which is a sinusoidal function with new

angular frequency  $\sqrt{\omega^2 - \frac{\gamma^2}{4}}$  (different from  $\omega$  which is called the natural angular frequency).



The above type of oscillation is called **underdamping**, where the mass still oscillates about the origin, but the amplitude decreases exponentially overtime.

**Case 2: when  $\frac{\gamma}{2} > \omega$ , the discriminant is positive**, which makes  $\alpha$  real. Due to the  $\pm$  sign, the general solution is

$$x = A_1 e^{(-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega^2})t} + A_2 e^{(-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega^2})t}$$

$A_1, A_2$  depends on the initial conditions, such as its initial velocity, position etc. But nonetheless, do check that both terms have negative powers and approaches zero as time increases. This is called **overdamping** where the mass do even oscillate but just approaches equilibrium position.

**Case 3: when  $\frac{\gamma}{2} = \omega$ , the discriminant is zero**, which makes  $\alpha$  real. The solution seems to be  $x = Ae^{-\frac{\gamma}{2}t}$  (it is not very complete as the motion may differ due to different initial velocity and position etc, but let's just accept it for now). I like to think of this as a special case of overdamping, where the value for the power is the largest in magnitude, which also means the exponent decays the fastest. This is **critical damping**, where the system returns to equilibrium position in the shortest time possible.

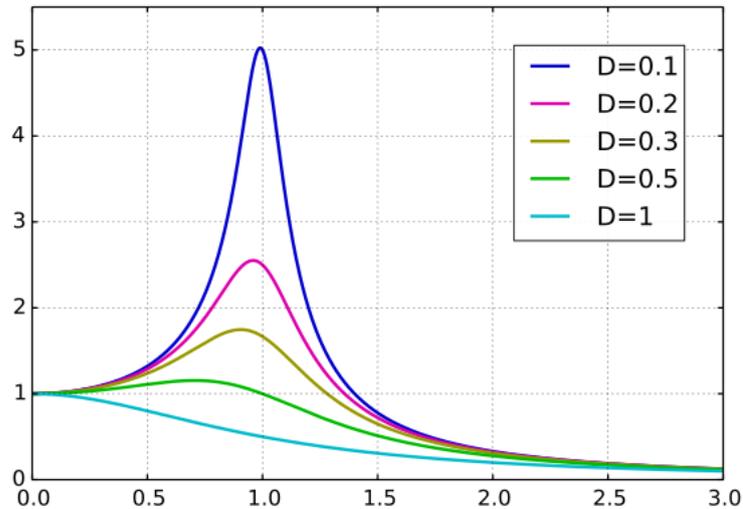
Just to summarise, depending on the value of  $\gamma$  and  $\omega$ , the type of damped oscillation may differ. When  $\gamma$  is relatively low, the damping force is relatively small and the system undergoes **underdamping**, where the system still oscillates but the amplitude decreases over time. When  $\gamma$  is relatively high, the system undergoes **overdamping**, where the system no longer oscillates but just returns to equilibrium position. In between the two types, there is **critical damping**, where the system returns to equilibrium in the shortest time.

## 5.2 Forced Oscillation

I'll avoid the Math in this section as I find it really difficult.

Apart from damping forces, there can also be driving forces. It typically refers to a periodic driving force like  $F = \cos(\omega_d t)$ .  $\omega_d$  is the frequency of the driving force, which we can differ from  $\omega_0$ , the natural frequency of the system.

This forces the system to also oscillate at  $\omega_d$  in the long run, but there may be a phase difference between the driving force and the oscillation of the system (this means they are at different points of the sine graph).



[https://commons.wikimedia.org/wiki/File:Mplwp\\_resonance\\_D.svg](https://commons.wikimedia.org/wiki/File:Mplwp_resonance_D.svg)

The diagram shows the graphs of amplitude against the driving frequency. The different graphs are due to different extent of damping. The amplitude of the driven oscillation will differ based on the driving frequency.

Under light damping, a phenomena called **resonance** occurs, where the maximum amplitude occurs when the driving frequency is close or equal to the natural frequency of the system.

As damping increases, the amplitude of oscillation decreases, the peak becomes less sharp and is shifted slightly to the left. In the ideal case where there is no damping, the amplitude will reach infinity at exactly the natural frequency.

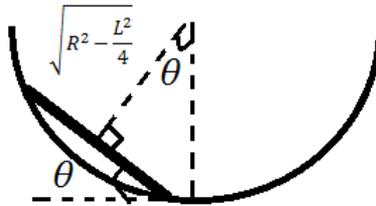
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**O2. (Pan Pearl River Delta Physics Olympiad 2005)** A uniform rod of length  $L$  and mass  $M$  is resting in a smooth hemisphere of radius  $R(> 0.5L)$ , as shown. Find the vibration frequency of the rod about its equilibrium position. ( $\cos\theta \approx 1 - \frac{\theta^2}{2}$ )



I decided to end off the chapter with a problem, which did not simply just give you the equation for the potential energy. It requires a bit of derivation from the physical system itself.

It should be quite obvious that the equilibrium is as shown in the diagram above, when the rod is horizontal at the bottom of the hemisphere. The hemisphere is smooth, which means we can neglect energy loss due to lack of friction. The question probably requires one to bring in torque and everything if we were to use forces, so let's try to solve it using energy.



I'll define a variable  $\theta$  as the angle the line connecting centre of mass of the rod and the centre of the sphere makes with the vertical as shown in the diagram above. Let's consider all the various energy in the system.

$$\begin{aligned}
 E &= GPE + KE \\
 &= GPE + KE_{translational} + KE_{rotational} \\
 &= -Mg\sqrt{R^2 - \frac{L^2}{4}}\cos\theta + \frac{1}{2}M(R^2 - \frac{L^2}{4})\dot{\theta}^2 + \frac{1}{2}(\frac{1}{12}ML^2)\dot{\theta}^2
 \end{aligned} \tag{2}$$

If I were to use the approximation  $\cos\theta \approx 1 - \frac{\theta^2}{2}$ ,

$$\begin{aligned}
 E &= -Mg\sqrt{R^2 - \frac{L^2}{4}} + \frac{Mg}{2}\sqrt{R^2 - \frac{L^2}{4}}\theta^2 + \frac{1}{2}M(R^2 - \frac{L^2}{4})\dot{\theta}^2 + \frac{1}{2}(\frac{1}{12}ML^2)\dot{\theta}^2 \\
 k &= Mg\sqrt{R^2 - \frac{L^2}{4}} \quad m = M(R^2 - \frac{L^2}{4}) + \frac{1}{12}ML^2
 \end{aligned}$$

As you can see, I'm left with a constant, a term quadratic in terms of  $\theta$  and a term quadratic in terms of  $\dot{\theta}$ , which is really similar to our EPE and KE equation. Using the formula for angular frequency, we can then calculate the vibrational frequency as  $\frac{1}{2\pi}\sqrt{\frac{gh}{h^2 + \frac{1}{12}L^2}}$ , where  $h = \sqrt{R^2 - \frac{L^2}{4}}$ .