

# Gravitation and Celestial Mechanics

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## 1 Introduction

Gravitation is a simple idea: two objects with mass appears to display an attractive force. And when there is a force, Physicists would like to find out about it. According to my knowledge, the formula we are learning is not the most powerful formula, but it is good enough.

Before we start, let me inform you that this chapter heavily depends on concepts in previous chapters. Likewise, we will look at this from two perspectives: force and energy.

## 2 Newton's Law of Universal Gravitation

This law merely draws a relation between various related quantities: the gravitational force, the mass of the two objects, and the distance between the two objects.

**The force due to gravity between two point objects is directly proportional to the product of their masses and inversely proportion to the square of the distance between them.**

$$F = G \frac{m_1 m_2}{r^2}$$

$G$  is the universal gravitational constant, usually given at the front of your exam paper. Like any force, it obeys Newton's Third Law of Motion.

Like any force, we need to know the direction. Gravity is an attractive force. For the first mass, the force points towards the second mass. For the second mass, the force points towards the first mass.

Everything seems logical. The greater the mass, the greater the force. When someone has a greater mass, one would need a greater force to jump. This is because he needs to exert a greater force to overcome the gravitational force.

The further the distance, the weaker the force. Think about outer space, where the gravitational force seems to become weaker.

Previously, we have assumed that the gravitational force for an object is constant on Earth, regardless of how high you are. It is not exactly true, but it is a good approximation. If the distance from the surface of Earth is small compared to the radius of the Earth, the difference in the value of  $r$  and the value for  $F$  is insignificant.

### 3 Gravitational Fields

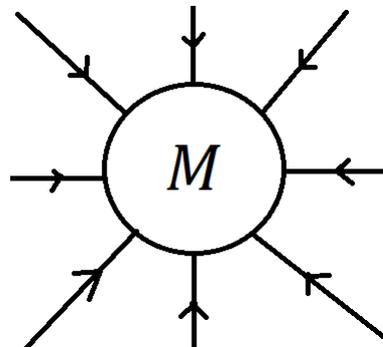
Consider a system containing a planet like Earth with mass  $M$ . Regardless of what is the mass of the object, based on Newton's Universal Law of Gravitation, the force will be given by:

$$F = \left( G \frac{M}{r^2} \right) (m)$$

If I were to replace the term  $G \frac{M}{r^2}$  by  $g$ , the force on any object with a known mass,  $m$ , is just  $mg$ .  $g$  is a vector quantity (in the same direction as the gravitational force) known as the gravitational field. It tells us the gravitational force per unit mass. Do note that people always differentiate this from acceleration due to gravity (even though they give the same value and units).

A question may arise: most planets and stars are not exactly a single dot in space. So where should the value for  $r$  start from? For our purpose, we usually assume planets to be spherical, and we can assume it to be a point mass positioned at the centre of the sphere.

#### 3.1 Visual Representation



Often, gravitational fields are represented as arrows or lines as seen in the diagram above. The direction of the arrow represents the direction of the field, which also gives the direction of the force of a mass placed at that position. As you move further away from the mass, the field lines are also more spread

out(further from each other), which indicates that the strength decreases as distance increases.

Like any vector, gravitational field can also undergo vector addition to give the net gravitational field due to multiple sources.

Anyways, the gravitational field at a distance  $r$  away due to a point source with mass  $M$  is given by  $G\frac{M}{r^2}$

## 4 A Nice Result

### 4.1 Gauss's Law for Gravity

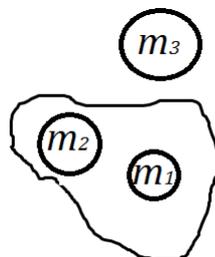
There is something special about a force or field that is inversely proportional to distance squared. I believe it has to be due to the fact that the surface area of a sphere is directly proportional to its radius squared(surface area =  $4\pi r^2$ ).

Consider a point object with mass  $M$ . It creates gravitational field given by  $G\frac{M}{r^2}$ . Let's imagine an imaginary spherical shell of radius  $r$  with the point object at the centre. At every part of the shell, some gravitational field will be passing through it.

We shall consider a new quantity defined by the product of the gravitational field passing through the surface **perpendicularly**, and the area of the surface(we shall call this quantity gravitational flux). It is quite obvious that in our case, the total gravitational flux for the spherical shell is given by

$$\Phi = \left(-G\frac{M}{r^2}\right) (4\pi r^2) = -4\pi GM$$

Here we have Gauss's Law for gravity. Suppose I have an enclosed 3D surface(think of it as a balloon with no holes) which has a total mass of  $M$  enclosed in it. The total flux going through the surface will be given by  $-4\pi GM$ . Apparently this is true regardless of the shape of the balloon or the distribution of gravitational field on the balloon surface. The sign convention is such that negative is going into the balloon and positive is coming out of the balloon.



Let's consider the case in the diagram above. The surface is actually 3D but I hope you get the idea. The surface is very irregular and the distribution of gravitational field is probably not even. But it doesn't matter. I just have to add up the gravitational flux for every bit of the surface. The total should give me  $-4\pi G(m_1 + m_2)$ . The negative sign indicates that overall, there is greater gravitational flux entering than leaving the enclosed space.

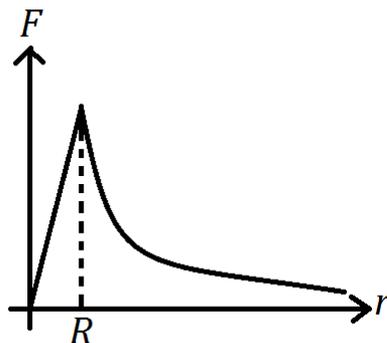
You may wonder if  $m_3$  contributes anything. Well, the simple answer is no. The precise answer is Mathematical but the basic idea is that while gravitational flux enters the balloon from one side of the surface, it leaves from the other. Overall, these two effects perfectly cancels out.

## 4.2 Shell Theorem

Using the idea of Gauss's Law, we can arrive at another result which can help us answer this question. Suppose you are inside of a spherically symmetric planet, what will be the gravitational force that you will experience?

Suppose you are a distance  $r$  away from the centre. The force can be derived from the gravitational field, which can then be known using Gauss's Law. If I were to construct a spherical shell of radius  $r$  that is concentric with the sphere, it would not be hard to realise that the gravitational field is equivalent to the case where only the part of Earth inside this shell present. This is due to the spherical symmetry. Let this digest for a moment.

Maybe an example will help. Consider Earth which we shall assume to be a perfect sphere with radius  $R$  and density  $\rho$ . What force will someone with mass  $m$  experience as  $r$  (distance from the centre of Earth) varies? Consider a few cases. First when  $r < R$ . We shall make use of our previous result. The force is just  $G \frac{(\frac{4}{3}\pi r^3 \rho)m}{r^2} = G \frac{4}{3}\pi r \rho m$ . Interestingly, the force increases linearly with  $r$ . Well,  $R \leq r$  is simple, its just as per normal:  $G \frac{(\frac{4}{3}\pi R^3 \rho)m}{r^2}$ . So this is how the gravitational force,  $F$ , varies with  $r$ .



Let's consider the most extreme case, where  $r = 0$  (ie you are at the centre of Earth). From the graph, we know that  $F = 0$ . This seems logical, because

you experience the same force from all directions. Overall, net force is zero.

## 5 Gravitational Potential Energy

In previous chapters on forces and energy, I said that forces and energy are two sides of the same coin. Likewise, there is also an energy aspect of gravitation, which is basically the gravitational potential energy(GPE) which we are taught since primary school.

Instead of asking what is the energy at a point, we shall ask what is its difference with respect to a certain point, so don't be surprised if there is negative energy. For convenience and convention we shall assume potential energy at infinity(somewhere far far away) to be zero.

Let's denote the GPE in the case where there is a human with mass  $m$  a distance  $r$  away from a planet with mass  $M$  to be  $U(r)$ . I'll need to exert a force and therefore do some work in order to bring it further from the planet to infinity. Let's suppose it always move in the same velocity and thus have a constant kinetic energy.

This work done acts as energy input into the system. Overall, I have an equation that looks like this.

$$U(r) + W = U(r) + \int_r^\infty G \frac{Mm}{r^2} dr = U(\infty) = 0$$

For those who solved the integration, you should get  $U(r) = -G \frac{Mm}{r}$ . Do take note of the negative sign.

You may ask who has the energy? Is it the human or the planet? Actually it's both. The energy is the energy of the planet-human system. The energy would not have existed without either the human or the planet.

So what if there are more than 2 objects. We just have to find all combinations of 2 objects and add up the corresponding GPE. Let's say there are 3 objects A,B and C. I just have to find the GPE for AB, AC and BC.

### 5.1 Gravitational Potential

Similar to gravitational force, which has a concept of force per unit mass, there is a similar concept for GPE, that is called gravitational potential. Given a

planet with mass  $M$ , it will create a region with gravitational potential given by  $\phi = -G\frac{M}{r}$ . When a mass  $m$  is present, the GPE is then  $m\phi$ .

## 6 Connection to Other Topics

Like I have said, this topic is heavily dependent on other topics. For the sake of simplicity, we shall use the system of a planet with mass  $m$  orbiting around a much heavier star with mass  $M$ . I say much heavier so that I can make an assumption, which I'll explain in a later section.

### 6.1 Conservation of Angular Momentum

This is a very useful principle which we can apply in celestial mechanics. In gravitation, we know that the force always acts along the line connecting the two objects (such a force is known as a central force). As we know, such a force will not generate a torque and thus will not lead to a change in angular momentum,  $L$ .

### 6.2 Conservation of Energy

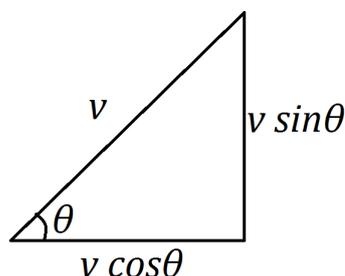
For simplicity of what we are learning, there shall be nothing that can lead to energy loss or gain in space.

GPE is dependent on position and not velocity. Therefore, as long as I know the distance between two objects, I can know the GPE.

In such systems, we are only concerned in the kinetic energy and GPE, and I know the sum of these two energy does not change with time due to the law of conservation of energy.

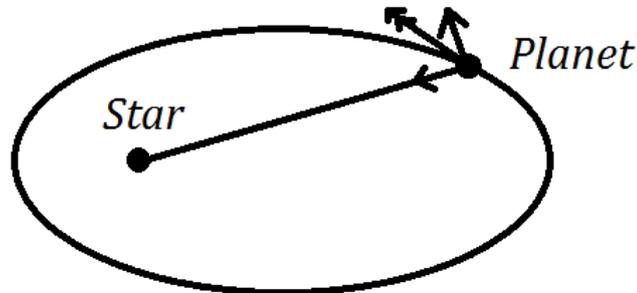
$$\frac{1}{2}mv^2 - G\frac{Mm}{r} = E$$

We are always talking about two bodies moving in the same plane (is there a reason?). Although kinetic energy is a scalar quantity, Mathematically, it is still the sum of the kinetic energy of two perpendicular components. This is due to Pythagoras Theorem.



$$\frac{1}{2}mv^2 = \frac{1}{2}mv^2 \cos^2\theta + \frac{1}{2}mv^2 \sin^2\theta$$

In the case of celestial mechanics, it is probably convenient to resolve into two perpendicular components: one along the line connecting the planet and star, and one perpendicular to it.



$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - G\frac{Mm}{r}$$

A variable with a dot on top just means its derivative with respect to time (the rate of change with respect to time). This equation is rather troublesome as I have two variables:  $r$  and  $\theta$ . However, it is known that they are related by the expression for angular momentum:  $L = rp = mrv = mr^2\dot{\theta}$ , which fortunately, is a constant. Therefore, with some manipulation, I get:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - G\frac{Mm}{r}$$

As we can see, there is only one variable  $r$  left. A lot of times, we talk about motion in one dimension. Our star-planet system is physically not one dimensional but I can sort of think of it as one dimensional if I just consider movement along  $r$ . In that case, my effective potential energy can be thought of as:

$$V_{eff}(r) = \frac{L^2}{2mr^2} - G\frac{Mm}{r} \quad E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$$

A lot of problems involve solving this equation. Some may even involve a potential energy not governed by the formula  $-G\frac{Mm}{r}$ , but the idea is the same.

## 7 Kepler's Laws

This section is adapted from <http://hyperphysics.phy-astr.gsu.edu/hbase/kepler.html> as well as Morin's Introduction to Classical Mechanics.

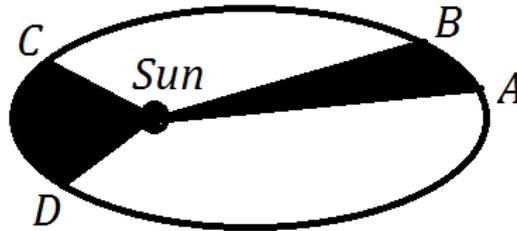
### 7.1 1st Law: The Law of Orbits

If we were to solve the conservation of energy equation above, it turns out that the motion of a planet around a star forms various conic sections (circle, ellipse,

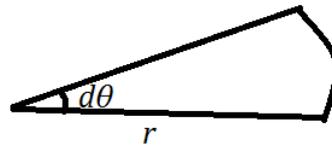
parabola, hyperbola), depending on various conditions. In the case of planets around the Sun, it is discovered that all planets move in elliptical orbits (think of it as a flattened circle), with the sun at one focus of the ellipse (not the centre).

## 7.2 2nd Law: The Law of Areas

The law states that: if I were to connect a line between the planet and the sun, it sweeps out equal areas in equal times.



So if the planet moves from A to B and C to D, both using the same amount of time, the two shaded parts have the same area.



I'll leave here a proof. Consider a small sector swept out by the planet. Even though it is an ellipse, for a really small angle,  $d\theta$ , it really isn't much different from a circle, which means the area swept is  $\frac{1}{2}r^2 d\theta$ . The rate of change of the area swept,  $A$ , with respect to time is thus

$$\frac{dA}{dt} = \frac{1}{2}r^2 \frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta}$$

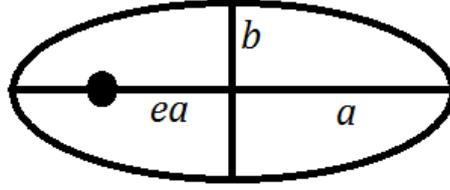
since  $r$  is constant over this small angle. If you realise, the formula looks similar to that of angular momentum, which is a conserved quantity.

$$\frac{dA}{dt} = \frac{mr^2\dot{\theta}}{2m} = \frac{L}{2m}$$

Therefore, the rate of change of area swept is constant.

## 7.3 3rd Law: The Law of Periods

The period,  $T$ , is the time taken for the planet to make one full revolution, and there is a very nice result, partly due to the Mathematics of an ellipse: the period for an ellipse with semi-major axis,  $a$ , is equal to the period of a circle with radius  $a$ .



Well, for a circle, the "semi-major axis" is just its radius. The gravitational force is what gives rise to the centripetal force that allows the planet to undergo circular motion.

$$G \frac{Mm}{a^2} = m \frac{v^2}{a} = ma\omega^2 \implies \omega = \sqrt{\frac{GM}{a^3}}$$

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{a^3}{GM}} \implies T^2 = \frac{4\pi^2 a^3}{GM}$$

Surprisingly, the shape of the ellipse doesn't matter.  $e$  in the diagram is a variable called the eccentricity, which tells us the shape of a conic section (circle, ellipse, parabola, hyperbola). What is important is the semi-major axis.

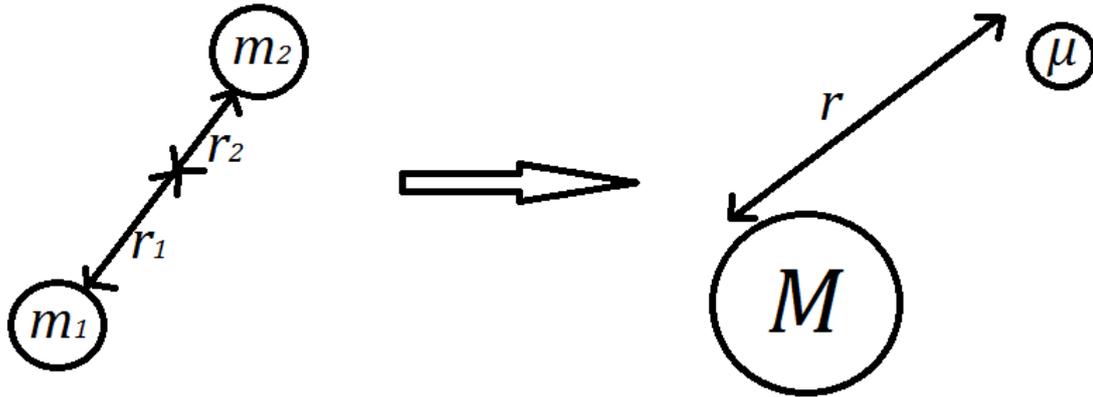
## 8 Reduced Mass

Previously I said I made an assumption about the star being much heavier than the planet. Until now, all our understanding revolves around a planet revolving around a stationary star. This is not exactly true. The star also experiences some force. However, it is a close enough approximation, as given the same force, the star experiences a much smaller acceleration and can be assumed to be stationary.

But now, one question arises, what if the two bodies have similar mass?

On one hand, I can calculate the forces for both bodies. On the other, I can find the equivalent mass such that the whole system is sort of equivalent to the model where a planet is revolving around a stationary star, in terms of both forces and energy.

Suppose there are two bodies with mass  $m_1$  and mass  $m_2$ ,  $r_1$  and  $r_2$  away from the common centre of mass respectively. We shall model this as a planet with mass  $\mu$  revolving around a stationary star with mass  $M$  with a distance  $r$  between them.



$$F = G \frac{m_1 m_2}{(r_1 + r_2)^2} = G \frac{M \mu}{r^2}$$

$$KE = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 = \frac{1}{2} \mu r^2 \omega^2$$

$$GPE = -G \frac{m_1 m_2}{r_1 + r_2} = -G \frac{M \mu}{r}$$

(Can you explain to yourself why the two masses have the same angular velocity?)

The solution that I got is  $M = m_1 + m_2$ ,  $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}}$ ,  $r = r_1 + r_2$ . The formula for  $\mu$  is that for reduced mass.

Let's look back at our original assumption. The Sun is approximately 333,000 the mass of Earth. The mass of Sun is definitely considered much larger than that of Earth. Let's suppose  $m_1, m_2$  be the mass of Sun and Earth respectively.  $m_1$  is much bigger than  $m_2$ ,  $m_1 \gg m_2$ .  $m_1 + m_2$  is really not much different from  $m_1$ . If I were to do the approximation for the formula for  $M$  and  $\mu$ , I get  $M = m_1 + m_2 \approx m_1$ ,  $\mu = \frac{m_1 m_2}{m_1 + m_2} \approx \frac{m_1 m_2}{m_1} = m_2$ , which is exactly what we have been doing before this section on reduced mass. Hope this gives you a sense of relief that you don't have to unlearn everything previously.