Rotations 1

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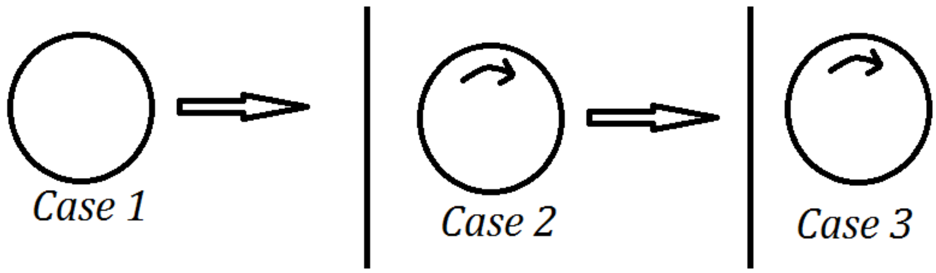
1 Introduction

In Kinematics, we've said that there are two types of motion: linear and rotational. Similarly, there is an equivalent concept for rotational motion in dynamics.

This is relevant as long as we are talking about an object which has a volume (not just a point), and I do believe this makes up most of the objects we see in this world. As long as an object has a volume, it will have the capability to rotate.

These three cases convinced me why rotation deserves a chapter on its own(like most textbooks do).



In case 1, the ball is merely translating rightwards, not rotating, while in case 2, the ball is both translating and rotating, while is case 3, the ball is only rotating. A lot of times, we have to consider the rotation and translational aspects separately.

This set of notes will focus on delivering content on the introduction to rotations.

2 Rigid Bodies

For most of what we discuss, we be referring to rigid bodies. It is the idea that the distance between two particles remains constant. A lot of things in real life can be considered as rigid bodies, such as your phone, chair and iPad, until you decided to bend, break or crack them. Certain things are not rigid body. For instance, as you swing a piece of dough, it tends to elongate as you swing.

In general, there are two conditions for static equilibrium.

1. There is no **net** force acting on the body.

2. There is no **net** torque acting on the body **about any point**.

If you see, it is really about there being nothing to affect the translational and rotational aspects of motion.

3 Relationship between Linear and Angular Quantities

For all angular quantities, we can find an analogy with a linear quantity and most formula will look exactly the same. The table below shows these quantities and their **typical** symbols.

Linear Quantity	Angular Quantity
Displacement, s	Angular Displacement, θ
Velocity, v	Angular velocity, ω
Acceleration, a	Angular acceleration, α
Mass, m	Moment of inertia, I
Force, F	Torque, τ
Momentum, p	Angular Momentum, L
Translational Kinetic Energy	Rotational Kinetic Energy

We have seen the first 3 quantities in kinematics and it should be rather intuitive why they work in exactly the same way as their corresponding linear quantities. We will mainly focus on the others.

Do keep in mind that torque, angular momentum, angular velocity and angular acceleration are vectors/can be considered vectors. But for our purpose, I'll try not to involve the vector aspects.

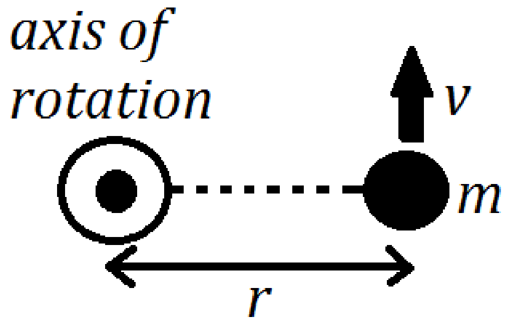
4 The General Idea

When discussing rotation, we need to know the axis of rotation, that is an imaginary infinite line which the object rotates about. Consider a rolling ball, the axis of rotation goes through the centre of the ball.

Previously on **forces**, we explain forces as a quantity that accelerates/decelerates an object(i.e. changes an object's **momentum**). The extent of this acceleration depends on a quantity called **mass**. Similarly, we will come across a quantity called **torque** that causes angular acceleration(i.e. changes something called the angular momentum).

4.1 Angular Momentum

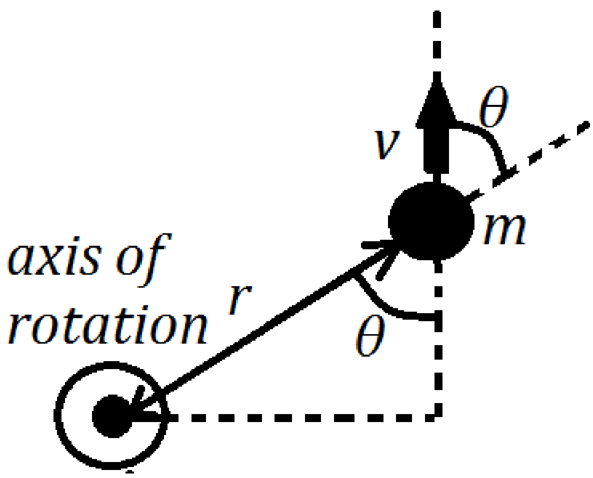
I'll try to explain everything from first principles: Newton's Laws of Motion.



Consider an object of mass m travelling at velocity v a distance r away from an axis of rotation. At this instant, it can also be thought of as rotating about the axis of rotation with angular velocity:

$$\omega = \frac{v}{r}$$

Now consider what happens sometime later if there are no external forces acting on the object. By Newton's 1st Law of Motion, it would travel in the same direction at the same velocity.



In terms of rotation about the axis of rotation, what quantity is being conserved? r changes, but $r\ sin\theta$ does not. So we can see that the following quantity is conserved.

$$L = mvr\ sin\theta = rp\ sin\theta$$

This quantity is the angular momentum, denoted by L. p is the momentum. I'll avoid using vectors, but just note that angular momentum is more precisely calculated by

$$\vec{L} = \vec{r} \times \vec{p}$$

4.2 Torque

Now, let's consider a force F being exerted in the direction of motion of the object. Using Newton's 2nd Law of Motion, we know that it will experience a change in momentum and thus, also a change in angular momentum. At that instant which the force is exerted, $r\ sin\theta$ remains constant, therefore, the rate of change of angular momentum is given by:

$$\tau = \frac{dL}{dt} = rF\ sin\theta$$

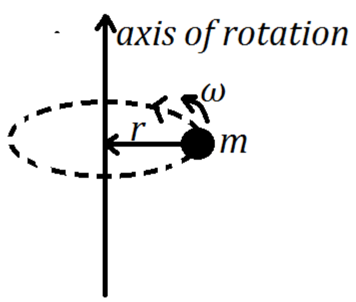
This quantity is known as the torque. I believe there are some subtle differences between torque and moments(which is what most of us learnt in school), but I like to treat them as equivalent. Just remember it as the product of the force and the perpendicular distance between the point which the force is exerted and the reference point.

Torque is actually a vector quantity, which means direction is important. It's logical since a force can either slow down or speed up the rotation of an object, depending on whether it is turned clockwise or anticlockwise.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

4.3 Moment of Inertia

Here we shall focus on rigid bodies. About the axis of rotation, the velocity of the particle is perpendicular to the line connecting the object and the axis of rotation.



Looking back at the formula for angular momentum. Since $\theta = 90°$ and $v = r\omega$,

$$L = rp\ sin\theta = rp = rmv = mr^2\omega$$

Likewise, if a force is exerted perpendicular to the line connecting the object and the axis of rotation, it will experience a torque, that is a change in angular momentum.

$$\tau = mr^2\alpha$$

α is the angular acceleration.

From the above two formula, you may realise the mr^2 term. This is a quantity related to an object's tendency to rotate, or rather to resist changes in its rotational motion. This is similar to mass, where a larger mass means a larger force is needed to slow down or speed up the object.

For a point mass of mass m at a distance r away from the axis of rotation, the moment of inertia of that mass about the axis of rotation is:

$$I = mr^2$$

If the system is made up of a lot of points, just add up the moment of inertia of each individual mass(or do an integration for a continuous mass distribution).

$$I = \sum mr^2 \quad I = \int r^2 dm$$

Given we know the moment of inertia about an object(about a certain axis of rotation), then we can calculate the angular momentum and know the angular acceleration when a torque is exerted. using

$$L = I\omega \qquad \tau = I\alpha$$

As you can see, it is really similar to the formula for linear momentum and force.

Let's calculate the moment of inertia for various simple objects. Hopefully it gives us a better understanding.

A Ring. Imagine a **very thin** ring of radius r and mass m. Its moment of inertia about its centre of mass(i.e. the centre of the ring) is rather intuitive. Every mass on the ring is at the same distance r from the centre of the ring. So if we were to add up $(\delta m)r^2$ for every single point mass on the ring, the total should give me mr^2.

A Circular Disc/Cylinder. These two cases are sort of identical, since the axis of rotation is a line extending to infinity. A disc is just a compressed cylinder(digest this before going on to the next sentence). Consider a cylinder of mass m and radius R. If I were to look at the cross-section, the mass per unit area is given by $\frac{m}{\pi R^2}$. A circle is made up of an infinite number of very thin rings. We just have to add up the moment of inertia of each individual ring centred at the axis of rotation.

The moment of inertia for each extremely thin ring of radius r and radius δr is given approximately by

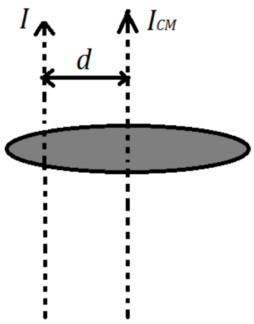
$$\delta I = 2\pi r \, \delta r \, \left(\frac{m}{\pi R^2}\right) r^2$$

This is basically the formula for the moment of inertia of a ring. The formula will be exact if the ring has a "zero" thickness. Now we just have to add up the moment of inertia for each individual ring, by integration.

$$I = \int dI = \int_0^R 2\pi r \, \left(\frac{m}{\pi R^2}\right) r^2 \, dr = \frac{2\pi m}{\pi R^2} \left[\frac{1}{4} r^4\right]_0^R = \frac{1}{2} m R^2$$

A Thin Rod I'll use this to explain a new concept: parallel axis theorem, as well as a technique called scaling.

Suppose an object of mass m has a moment of inertia of I_{CM} about a specific axis through its **centre of mass**, then what is its moment of inertia I about an axis that is parallel to the original axis, at a perpendicular distance d away?



The Math may look a little intimidating but it is actually quite simple. Let the two parallel axes that we are talking about be parallel to the z axis. We shall talk about an extremely thin object with its centre of mass at the origin(since any object can be compressed parallel to the z axis to become a pancake, just that the mass distribution will become weird) and this will not affect the moment of inertia.

Each slice will lie in the $x - y$ plane. By Pythagoras Theorem, the moment of inertia of that slice about the original axis is just

$$I_{CM} = \int r^2 \, dm = \int (x^2 + y^2) \, dm$$

Suppose the distance d is along the x axis, the moment of inertia about the new axis is

$$I = \int ((x+d)^2 + y^2)\, dm = \int (x^2 + y^2)\, dm + d^2 \int dm + 2d \int x\, dm$$

On the right most expression, the 1st term is just I_{CM}. The second term gives md^2. The last term gives zero as it tells us the position of the centre of mass, which we said is at the origin where $x = 0$. Overall, it gives us

$$I = I_{CM} + md^2$$

Now to answering the question on the moment of inertia of a rod of mass m and length l through the centre of the rod. We can probably guess that the expression is of the form kml^2(the units work out), where k is just some constant we should find.

Using the parallel axis theorem which we have just learnt, we know that the moment of inertia about one end of the rod is $kml^2 + m(\frac{l}{2})^2$. Now we combine two of this rod together end to end to make a longer rod. The total moment of inertia will be doubled to give $2kml^2 + 2m(\frac{l}{2})^2 = (2k + \frac{1}{2})ml^2$. Doing this gives me a new rod of length $2l$ and mass $2m$ and the new moment of inertia we calculated is through its centre. If we equate both expressions:

$$(2k + \frac{1}{2})ml^2 = k(2m)(2l)^2$$

$$2k + \frac{1}{2} = 8k \implies k = \frac{1}{12}$$

And here we have it. The moment of inertia through the centre of a thin rod is $\frac{1}{12}ml^2$. Just to double confirm using Mathematics. Let $\lambda = \frac{m}{l}$ be the mass per unit length. The moment of inertia contribution due to a small segment δr of the rod a distance r away is

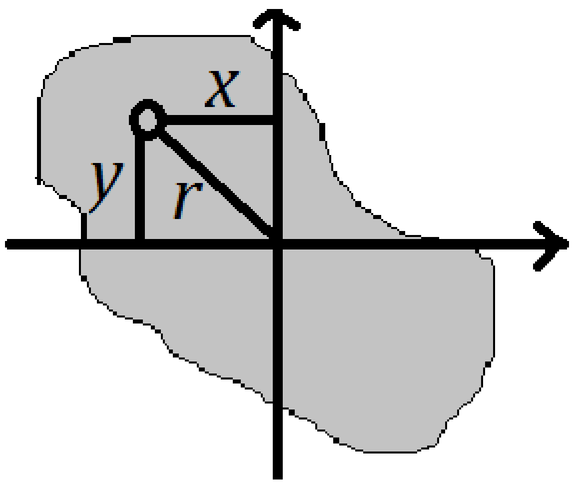
$$\delta I = \lambda\, \delta r\, r^2$$

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} \lambda r^2 dr = \lambda \left[\frac{r^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}} = \lambda \left(\frac{l^3}{24} - \left(-\frac{l^3}{24}\right)\right) = \frac{1}{12} ml^2$$

As expected, they give the same result.

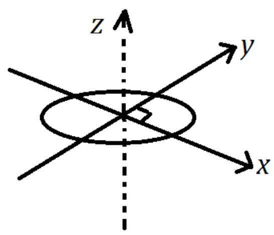
A Hoop A hoop is actually also a ring, but I am referring to its moment of inertia about another axis: an axis along its diameter. I'll use this to explain another new concept: perpendicular axis theorem.

Do note that this theorem requires a flat object, that is if you were to compress the object along an axis, the shape of the object will look exactly the same.



Consider this very oddly shaped, but flat object. I know the moment of inertia through an axis pointing out of the page in the centre of the 2 axes. It is just the sum of $\delta m\, r^2$ for every single small mass on the object. And by Pythagoras Theorem, I know that $r^2 = x^2 + y^2$. Therefore, the moment of inertia of a plane object about an axis perpendicular to the plane is equal to the sum of the moment of inertia about two axes perpendicular to each other that intersect at the point where the perpendicular axis passes through.

Now getting back to the question: what is the moment of inertia of the hoop with mass m and radius r about an axis of rotation along its diameter(either x or y).



By symmetry, I know that the moment of inertia about two diameters perpendicular to each other(x and y) are the same. And by the perpendicular axis theorem, they should add up to give the moment of inertia about an axis passing through the centre of the ring(ie z).

$$I_x + I_y = I_z = mr^2, \qquad I_x = I_y$$

$$I_x = \frac{1}{2}mr^2$$

5 Just to End Off

This set of notes is really content heavy. You may feel weird that someone came up with all these quantities that doesn't seem "fundamental" enough. But also ask this question: why did someone come up with the concept of force and mass?