Kinematics

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1 Introduction

Kinematics is the study of motion. When describing motion, we will ask questions on where it is (displacement), how fast it is moving (velocity) and how is its velocity changing (acceleration). We want to describe this motion using Mathematical equations.

2 Basic Definitions

I'll start off with the definitions. We are mostly interested in vectors, so we will be using displacement and velocity instead of distance and speed.

$$v = \frac{\Delta s}{\Delta t} \qquad a = \frac{\Delta v}{\Delta t} \qquad s = s_i + v\Delta t \qquad v = v_i + a\Delta t$$

where s is the displacement, v is the velocity, a is the acceleration, t is the time, and Δ just refers to a change in the quantity.

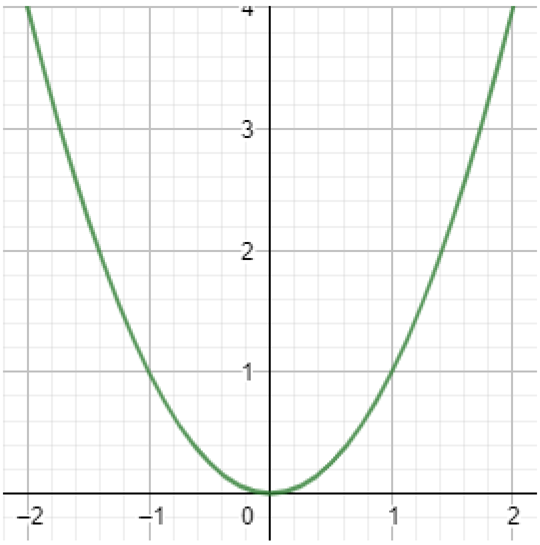
However, most of the time, the changes do not occur at a constant rate. We will need to involve calculus. Do see my previous note on Calculus.

$$v = \frac{ds}{dt} \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2} \qquad s = s_i + \int v\,dt \qquad v = v_i + \int a\,dt$$

3 Graphs

Alternatively, we can represent our displacement, velocity and acceleration as graphs, how it changes with time. With a 2D graph, we can only represent 1D motion: either forward or backward.

Let's use an example. An object's displacement follows the equation $s = kt^2$. k is just a constant with units $m\ s^{-2}$ (check that the units work out as we wanted). For simplicity, let's take $k = 1$. Therefore, I can represent its motion as a typical parabolic/quadratic curve:

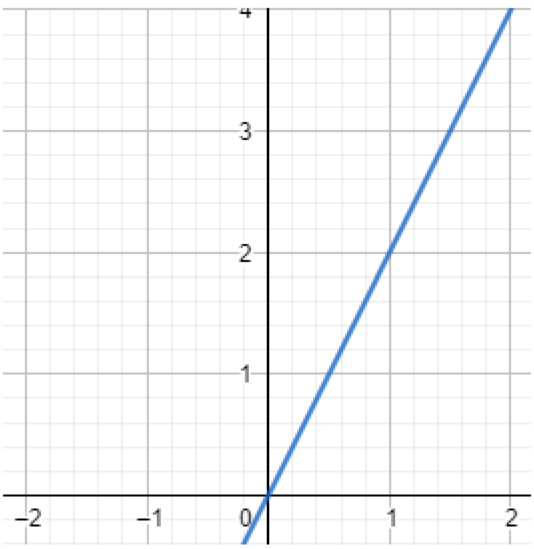


Graph of displacement against time. ($s = kt^2, k = 1$)

So, at $t = 0$, the object is at the origin, where displacement, $s = 0$. Then it starts moving in the positive direction at an increasing velocity. Intuition, as well as the formula for velocity, tells us that the velocity is simply the gradient(or derivative) of this graph.

$$v = \frac{ds}{dt} = \frac{d}{dt}kt^2 = 2kt = 2t$$

Therefore, my velocity-time ($v - t$) graph is a linear graph.



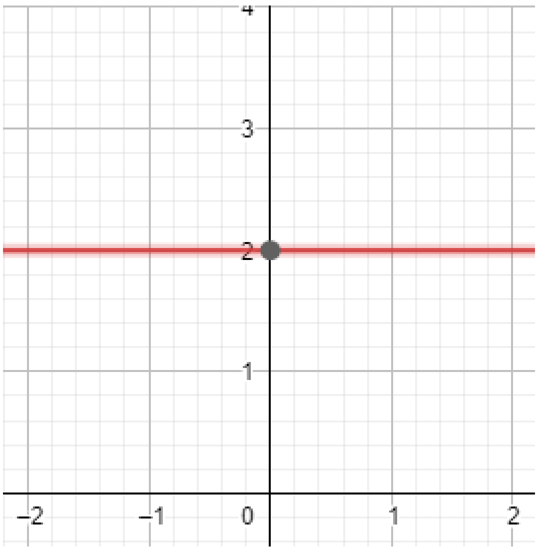
Graph of velocity against time. ($v = 2kt, k = 1$)

At $t = 0$, the object is not moving. Then, its velocity starts increasing linearly. The area below this graph should give me the change in displacement. Its gradient should give me the acceleration.

$$s = \int v\,dt = \int 2t\,dt = t^2 + C \text{ ,where C is an arbitrary constant}$$

Well, I know that when $t = 0, s = 0$. That means $C = 0$, which means $s = t^2$. Using the same way, I can find the acceleration.

$$a = \frac{dv}{dt} = \frac{d}{dt}2t = 2$$



Graph of acceleration against time. (a=2)

Graphs are just ways to help us understand motion. It can help us simplify some problems. I'll explain a few problems along the way. Before that, let's talk about kinematic equations.

4 Kinematic Equations

We need a few variables to fully describe an object's motion: displacement, velocity, acceleration, change in time. Given an object's velocity may change with time, we need to know both initial and final velocity.

However, usually, we don't have all of these variables. The equations are here to help us solve for one using the others. The equations help to relate the different variables. These equations only work if **acceleration is constant**. First and foremost, we have the easiest one:

$$v = u + at, \text{ where u is initial velocity and v is final velocity}$$

This is intuitive and useful if we only need to know the velocities without having to know the displacement.

$$s = v_{average}t = \frac{v + u}{2}t$$

This is also simple. Given acceleration is constant, the average velocity is just half of the sum of initial and final velocity. Its just like solving $1 + 2 + 3 + 4 +$

...... $+ 99$, $\frac{1+99}{2} = 50$ is the average of the series. There are 99 terms. Therefore the sum is $99 \times 50 = 4950$. Try drawing a graph to visualize this, if you realised, our previous formula looks exactly like the formula for finding the area of a trapezium.

Next, we can just substitute our first equation into our second equation.

$$s = \frac{v+u}{2}t = \frac{2u+at}{2}t = ut + \frac{1}{2}at^2$$

$$s = \frac{v+u}{2}t = \frac{2v-at}{2}t = vt - \frac{1}{2}at^2$$

The next equation is a little more troubling. Previously, we have created equations for cases where s, a, v, u are unknown respectively. Our next equation should be for when t is unknown. I don't know how, but let's lay down a few equations first.

$$v = u + at \Longrightarrow v - u = at$$

$$s = \frac{v+u}{2}t \Longrightarrow \frac{2s}{v+u} = t$$

Substituting the second equation into the first equation:

$$v - u = \frac{2as}{v+u} \Longrightarrow (v+u)(v-u) = v^2 - u^2 = 2as \Longrightarrow v^2 = u^2 + 2as$$

And here we have the last equation. Honestly do not try to memorize the equations, instead, try to practise more and it should enter your memory naturally. Let's try a few problems. Do try it yourself before looking at my answers.

K.1 *Consider a train that can speed up with an acceleration of* $20cm/s^2$ *and slow down with a deceleration of* $100cm/s^2$. *Find the minimum time for the train to travel between two stations* $2km$ *apart. You may assume that the train has to stop at every station. (SJPO 2010 General Round)*

Let's first consider this question: what are the constraints given in the question, and under these constraints, under what circumstances will the time be minimum?

Well, the first question is easy. I have my maximum acceleration and deceleration. I have a specified displacement ($2km = 2000m$), as well as the initial and final velocity, that is zero. Then, for the second question. It should be logical that the train should accelerate and decelerate at the maximum for the whole time. If not, I can simply accelerate/decelerate more, and the time taken should be lower.

Now, time to solve the problem. Let's call the maximum velocity attained to be v_{max}. I'll call the maximum acceleration and deceleration be a and d respectively, and $s = 2000m$. Logically, it would have accelerated for $\frac{v_{max}}{a}$ and it would have decelerated by $\frac{v_{max}}{d}$, since my initial and final velocity are zero. The total time for the journey should be:

$$t = \frac{v_{max}}{a} + \frac{v_{max}}{d} = v_{max}(\frac{1}{a} + \frac{1}{d})$$

Let's move on to another constraint, that is the overall displacement. We have this equation:

$$v^2 = u^2 + 2as$$

For us it is simple. One of the velocity is v_{max}, the other is 0. Therefore, the total displacement when accelerating is $\frac{v_{max}^2}{2a}$ and the total displacement when decelerating is $\frac{v_{max}^2}{2d}$. Therefore, the total displacement should be:

$$s = \frac{v_{max}^2}{2a} + \frac{v_{max}^2}{2d} = \frac{v_{max}^2}{2}(\frac{1}{a} + \frac{1}{d})$$

And we're done. We just have to substitute in $s = 2000m$, $a = 2.0 \times 10^{-1} m\ s^{-2}$, $d = 1.00 m\ s^{-2}$ into the two equations we got.

$$2000 = \frac{v_{max}^2}{2}(\frac{1}{2.0 \times 10^{-1}} + \frac{1}{1.00})$$

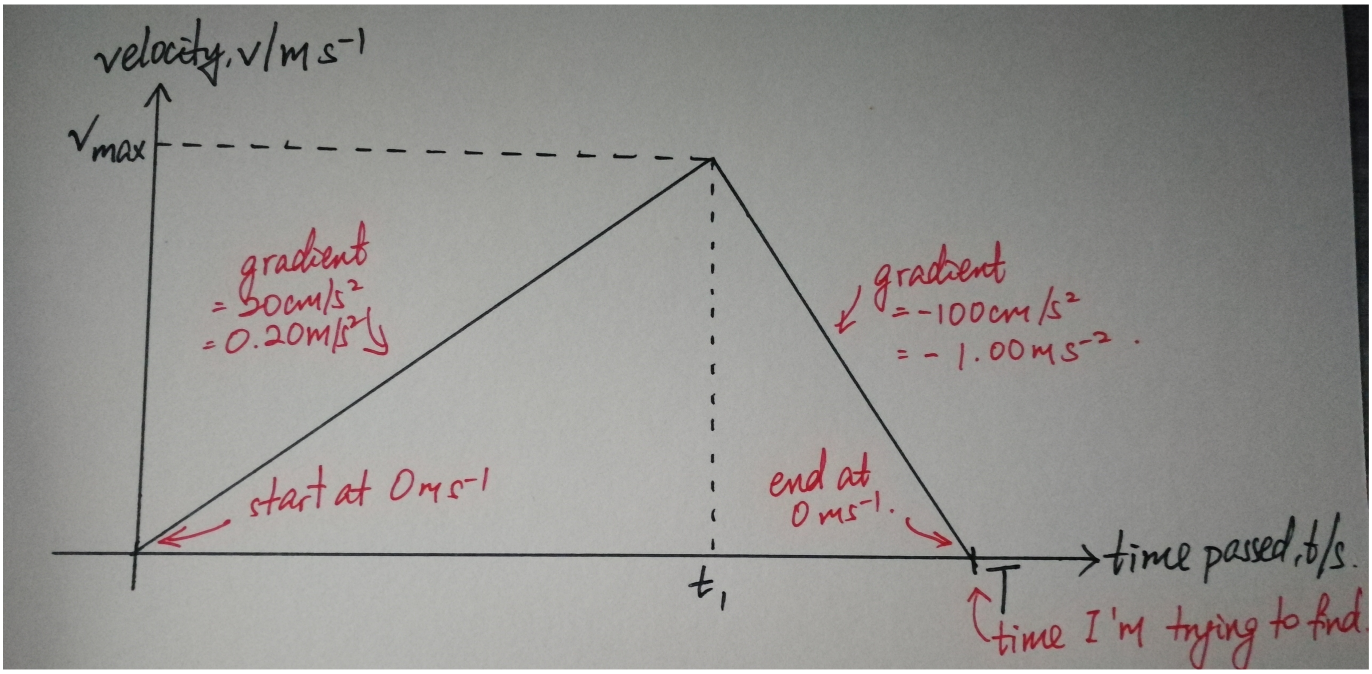
Therefore, $v_{max} = 25.8 m\ s^{-1}$. Substituting into the first equation we got:

$$t = v_{max}(\frac{1}{a} + \frac{1}{d}) = 25.8(\frac{1}{2.0 \times 10^{-1}} + \frac{1}{1.00}) = 155s$$

Honestly, when I first saw this problem in Secondary Three, all these Math would not come to my mind. However, there is an alternative way which is simpler to understand—draw the $v - t$ graph.

Why specifically the $v - t$ graph? Well, because I can attain information regarding both displacement and acceleration easily from a $v - t$ graph: displacement from the area below the graph, acceleration from the gradient of the graph.

Probably you'll a $v - t$ graph that looks like the one shown on the next page. I have annotated important information in red—either those that we need to use or those that we would like to extract from the graph.



Making use of the gradients, that is rise over run,

$$0.20 = \frac{v_{max}}{t_1} \qquad 1.00 = \frac{v_{max}}{T - t_1}$$

$$0.20t_1 = v_{max} \qquad 1.00T - 1.00t_1 = v_{max}$$

$$0.20t_1 = 1.00T - 1.00t_1 \Longrightarrow 1.20t_1 = 1.00T \Longrightarrow t_1 = \frac{5}{6}T$$

$$v_{max} = (0.20)(\frac{5}{6}T) = \frac{T}{6}$$

Yeah the dimensions in the above equation seem a little odd but we're only dealing with the numerical part. Next, the area below the $v - t$ graph should give me the displacement. Fortunately, it is a triangle, which area is easy to find.

$$2 \times 10^3 = \frac{1}{2}v_{max}T = \frac{T^2}{12}$$

$$T = \sqrt{12 \times 2 \times 10^3} = 155s$$

And we get to the same answer as the numerical method. Here is another problem for you to try as I'm too lazy to write the solution here.

K.2 *A train was moving with velocity* $96\,km\,h^{-1}$ *. It engaged its brakes for the first time at time* $99\ s$ *before coming to a halt. What was the distance at which it engaged the brakes? During the first* $\frac{1}{3}$ *of this distance, the train was decelerating with constant acceleration and decreased its velocity to* $\frac{1}{2}$*. During the second* $\frac{1}{3}$ *of the same distance, it was moving with constant velocity, and during the last* $\frac{1}{3}$*, it was again decelerating with constant acceleration until it stopped. (Online Physics Brawl 2018) (Ans: 1080m)*

5 1D + 1D = 2D

Up until now, we have only dealt with 1D kinematics, that is when the motion only occurs on one line. There is of course also 2D kinematics, that is when the motion occurs on a plane.

The reason I named this section this is because a lot of times, we just break up 2D motion into its 2 dimensions and analyze separately, using methods shown before. This is because displacement, velocity and acceleration are vectors, and vectors can be resolved into mutually perpendicular components.

Sometimes, we need to know which perpendicular components to break into. They can be the x and y axes, axes perpendicular/parallel to the object's motion, polar coordinates(a little more complicated) etc.

5.1 Relative Motion

Ever sat in a bus and see the commuters move pass you when in fact you (or the bus) are the one moving?

If A and B are moving at velocity $\vec{v_A}$ and $\vec{v_B}$ respectively, what velocity will A see B move at? Intuitively, it's $\vec{v_B} - \vec{v_A}$. Let's understand this in a vector diagram.

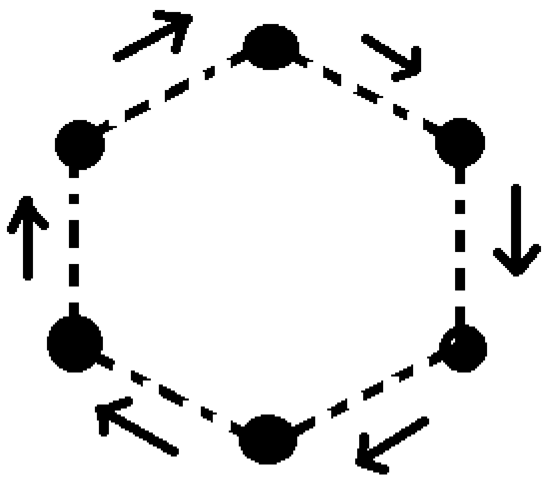
□Mechanics/Kinematics/relative.jpg

This work for any vector like displacement, velocity or acceleration. (Let $\vec{v_{i,j}}$ be the velocity of j as seen by i).

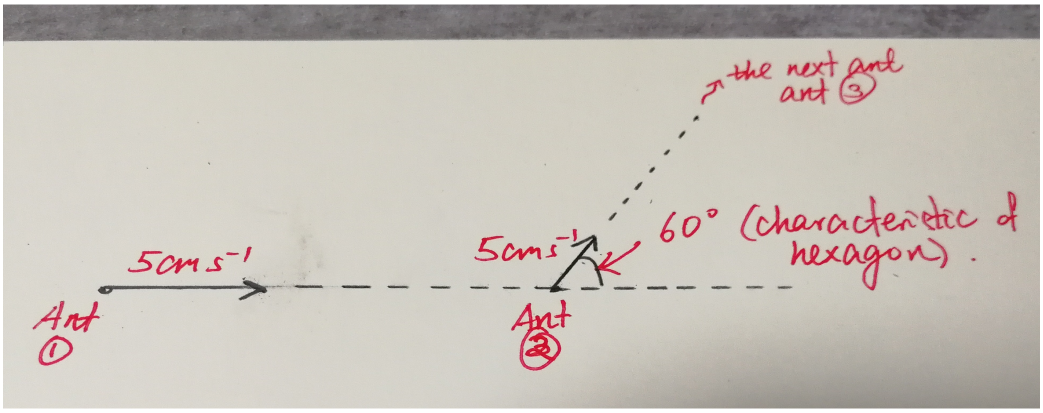
$$\vec{v_{1,n}} = \vec{v_{1,2}} + \vec{v_{2,3}} + ... + \vec{v_{n-1,n}}$$

K.3 *6 ants are each on a vertex of a hexagon with side length of 60 cm. They each set out towards the ant on the next vertex of the hexagon with a speed of 5 cm s^{-1}. How much time has passed before the ants meet each other? (XMPhysics PS go follow @xmphysics)*

It is difficult to visualize/lay out the equations to find out how each ant moves exactly, so let's try to simplify and understand the problem first.



How will the ants move afterwards? Well, the ants move at the same speed, each ant follows the next. Probably commonsense tells us: by symmetry, the ants will stay as a hexagon. At the end, the ants will be at the same point in the middle of the original hexagon. Let's look at two adjacent ants.



Since I know they will move in a hexagon, which decreases in side length, I just have to know the rate which this side length decreases with time and see how long does it take this length to reach zero. Probably, the rate at which each length decreases is just their relative speed along that line.

$$\text{relative speed} = 5 - 5cos60^\circ = 2.5cm\ s^{-1}$$

$$\text{time taken} = \frac{60cm}{2.5cm\ s^{-1}} = 24s$$

K.4 *Danka was watching a train with length* $l_1 = 120m$*, which was passing her with velocity* $v_1 = 60\ km\ h^1$*. She also saw a train of unknown length* l_2 *approaching from the opposite direction with velocity* $v_2 = 80\ km\ h^1$*. In order to find* l_2*, Danka measured the time the trains spent passing each other, i.e. the interval between the times when the fronts of the trains met and when the backs* *of the trains met. She measured $t = 9s$. How long was the second train?(Online Physics Brawl 2018)*

This question is relatively more straightforward and relatable. If you have taken an MRT before(EWL or NSL), you might have seen this situation where a train drive past your train, and it seems very fast, much faster than it seems when you are standing on the platform and a train drives past.

The two trains are moving towards each other at 60 and $80km\,h^{-1}$ respectively. Their relative speed is $60 + 80 = 140km\,h^{-1}$. Rephrasing what I said, to Danka, the second train is driving towards her at $140km\,h^{-1}$, while her train remains stationary. In her perspective, the other train moved by:

$$140km\,h^{-1} \times \frac{1h}{3600s} \times 9s = 350m$$

Let's look at the front of the moving train. It needs to move by the length of Danka's train, and it will be at the end of her train. Then, it will need to move by the length of the moving train itself, then the backs of the train will meet.

$$l_1 + l_2 = 350m \Longrightarrow l_2 = 350 - 120 = 230m$$

5.2 Projectile Motion

As the name suggests, this is the motion of a projectile ie when you throw an object into the air. Well, I haven't went through forces yet, but believe what I said, the object accelerates towards the ground(**downwards**) at a **constant acceleration**, that is g, usually given as $9.81\,m\,s^{-2}$. As long as I am close to the surface of Earth, the acceleration should be constant (Even for airplanes in the sky, the difference is insignificant. The difference will be discussed in gravitation).

Is there any horizontal acceleration? Well, we will assume negligible air resistance, so short answer, no. The horizontal velocity shall remain constant.

Let us have an object with velocity, v, projected with an angle, θ, above the ground, starting from the origin. It initial horizontal and vertical component of velocity are:

$$\text{horizontal velocity}, v_x = vcos\theta \qquad \text{vertical velocity}, v_y = vsin\theta$$

Then, the horizontal and vertical component of displacement, x and y, will be(refer to kinematic equations):

$$x = vtcos\theta \qquad y = vtsin\theta - \frac{1}{2}gt^2$$

Let's see how will the path look like. Doing some algebraic manipulation:

$$t = \frac{x}{vcos\theta}$$

$$y = vtsin\theta - \frac{1}{2}gt^2 = \frac{xsin\theta}{cos\theta} - \frac{1}{2}\frac{g\,x^2}{v^2cos^2\theta}$$

As we know, v, θ, g are all constants. Then the projectile path is just a parabolic/quadratic function.

K.5 *What angle should you launch an object to achieve the largest horizontal range?*

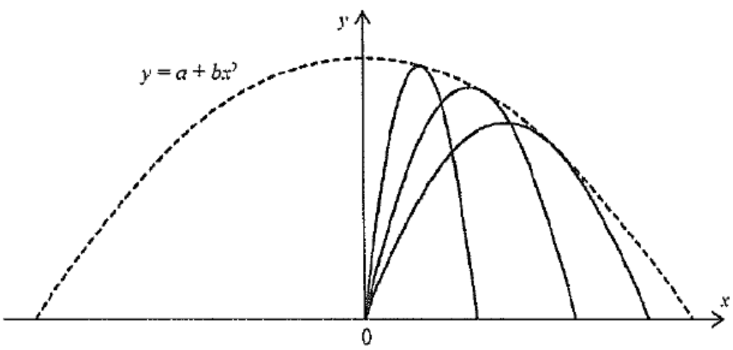
We shall use the equation that we got above. The horizontal range is just the distance the object flew before reaching the ground again. Basically, it is the other root of the quadratic equation, apart from $x = 0$.

$$0 = \frac{xsin\theta}{cos\theta} - \frac{1}{2}\frac{g\,x^2}{v^2cos^2\theta} \Longrightarrow \frac{sin\theta}{cos\theta} = \frac{1}{2}\frac{gx}{v^2cos^2\theta}$$

$$x = \frac{2\,sin\theta\,cos\theta\,v^2}{g} = \frac{v^2\,sin2\theta}{g}$$

A sine function ranges between 1 and -1. Its maximum is 1, when $2\theta = 90°$, $\theta = 45°$, which gives the maximum horizontal range as $\frac{v^2}{g}$.

K.6 *A projectile is launched at velocity v_0 into an ideal ballistic trajectory from the origin of a coordinate system. Given that: when the launch angle is varied, all the possible points that can be hit by the projectile are exactly contained within a parabola with equation $y = a + bx^2$ where y is the vertical height, x is the horizontal displacement from the origin, where a and b are constants. What could be the expression for a and b? (SJPO 2016)*



Let's first focus at solving the problem. The maximum height an object can reach is obviously when it is being thrown up vertically. That gives us the y-intercept as seen in the diagram, that is also a. Using a kinematic equation:

$$v_f^2 = v_i^2 + 2as \qquad a = -g,\; v_f = 0,\; v_i = v_0, s = \text{y-intercept} = a$$

$$0 = v_0^2 - 2ga \Longrightarrow a = \frac{v_0^2}{2g}$$

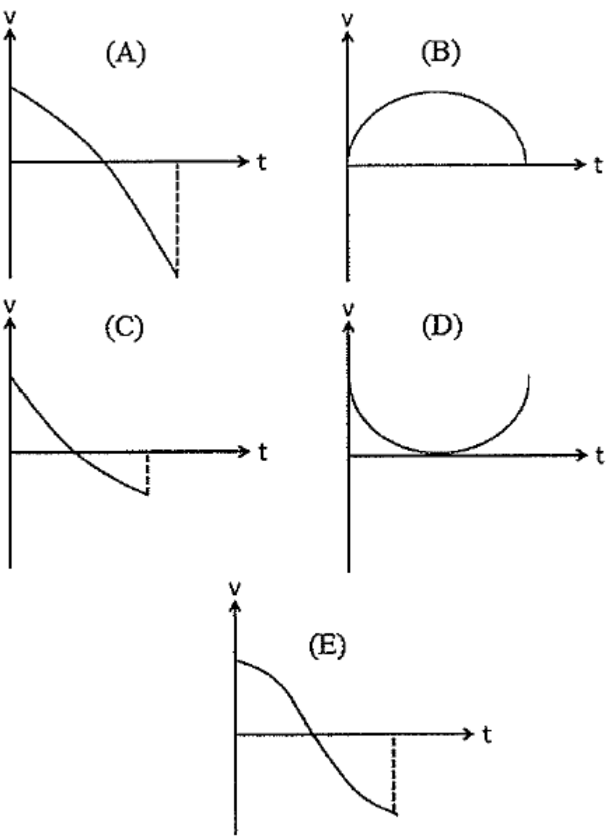
Now, let's look at the x-intercept. From K.5, we know they are the maximum horizontal range, that is $\frac{v_0^2}{g}$. Substituting into our original equation $y = a + bx^2$.

$$0 = \frac{v_0^2}{2g} + b(\frac{v_0^2}{g})^2 \Longrightarrow b = -\frac{g}{2v_0^2}$$

This is sufficient for answering the question in SJPO, where options are given.

Until now, we have been working under the assumption that air resistance is negligible, or absent, therefore, the object falls with a constant acceleration purely due to gravity. In reality, this is often not the case.
Air resistance is a force(I'll talk about this in detail in the next chapter but just accept that it is a source of acceleration), often proportional to velocity. Therefore, an object falling in air will one day reach constant velocity (called terminal velocity) ie no **net** acceleration as the acceleration downwards due to gravity is cancelled out by the acceleration upwards due to air resistance after it reaches that velocity.

K.7 *An object is thrown upwards from the ground and caught again when it fell back to the same place. Which of the following graphs most closely represent the velocity-time relationship during this process?(SJPO 2016)*



There are no numbers involved in this question. All I have here is qualitative explanation. This may be a little long-winded but I hope you understand my thought.

Given none of the options are linear, this means air resistance is probably not negligible. Therefore, the object experiences acceleration from two sources: air resistance and gravity. From experience or primary school science, air resistance acts in the opposite direction of motion. Therefore, the cases will be different when going up and coming down, so let's analyze them separately.

When going up, both these accelerations act downwards. As acceleration due to air resistance is proportional to velocity, the overall acceleration will decrease as the velocity of the object deceases as it goes up. At the top, it reaches zero velocity. Therefore, before *v=0*, the gradient should get less steep.

When coming down, the acceleration from air resistance is upwards, while the acceleration from gravity is downwards. As the object speeds up, the acceleration from air resistance increases, therefore, the overall acceleration decreases, and the gradient of the *v-t* graph becomes less steep. Until one day, the acceleration from air resistance perfectly cancels out the acceleration due to gravity, and the velocity becomes constant.

The graph that fits our explanation is **(C)**. A follow on question: what is the instantaneous acceleration when *v=0*?

5.3 Circular Motion

As implied by the name, circular motion occurs when an object moves in a circle. Just imagine tying a rock to a string, and then swing that string around. If you hold the string tight enough, the rock is undergoing circular motion. Before going into anything Physics, let's understand more Math about circles. You'll probably learn this in Math in secondary 4.

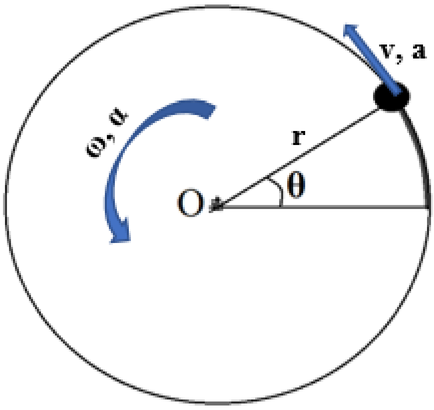
Radians is another way for measuring angles apart from degrees. You can write it without units, most people will understand, or you can write **rad** at the back.

Radians is defined as the ratio of the circumference of a sector of a circle (i.e. in the shape of a pizza slice) to its radius. Since we know the circumference of a full circle is given by $s = 2\pi r$, we know that $360° = 2\pi$, $180° = \pi$, $90° = \frac{\pi}{2}$ and so on......

What's significant about this is that it just simplifies a lot of things. For a sector, the circumference is just $r\theta$, area is just $\frac{1}{2}r^2\theta$.

5.4 Analogy to Linear Motion

If an object moves in a circle, after one whole round and returning to the same point, it would have no net displacement, velocity, or acceleration. But, it can have angular displacement (θ), angular velocity (ω), and angular acceleration (α).



Similarly,

$$\omega = \frac{d\theta}{dt} \quad \alpha = \frac{d\omega}{dt}$$

Since we know the circumference which the object moved is $r\theta$, we know that its linear velocity and acceleration are given by:

$$v = \frac{d}{dt}r\theta = r\frac{d\theta}{dt} = r\omega \quad a = \frac{d}{dt}r\omega = r\frac{d\omega}{dt} = r\alpha$$

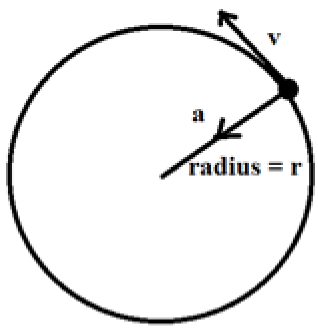
Note: this is referring only to the tangential component, the velocity and acceleration along the circumference of the circle.

5.5 The Radial Component

If an object moves around in a circle at constant velocity, is it experiencing acceleration? Of course. Velocity is a vector quantity, and given its direction is changing, it means it is experiencing a change in velocity and therefore an acceleration.

Thinking about the scenario of a swinging rock. The string is pulling the rock inwards, to the centre of the circle. Therefore, it is natural to believe that an object undergoing circular motion experience an acceleration towards the centre of the circle, that is the radial component.

Let's say an object is moving in a circle of radius r at a velocity v, it is experiencing a **net acceleration** in the radial direction. This acceleration is called the **centripetal acceleration**. Note that the velocity and radius may be changing at every point on the "circle", but the formula should still work.



$$a = \frac{v^2}{r} = r\omega^2$$

This formula may not appear intuitive. There is a proof, but I am unable to understand it when I was in High School, so let's just accept it (I've attached a proof in the appendix). But let's just look at the units to know whether the formula is even possible. v has units $m\,s^{-1}$. r has units m. Therefore, $\frac{v^2}{r}$ has units $m\,s^{-2}$, which is exactly the units for an acceleration.

K.8 *With what velocity should you run at the equator in order to weigh as much as possible (e.g. have the maximum possible weight, not mass) if you can choose the optimal direction?(Online Physics Brawl 2017)*

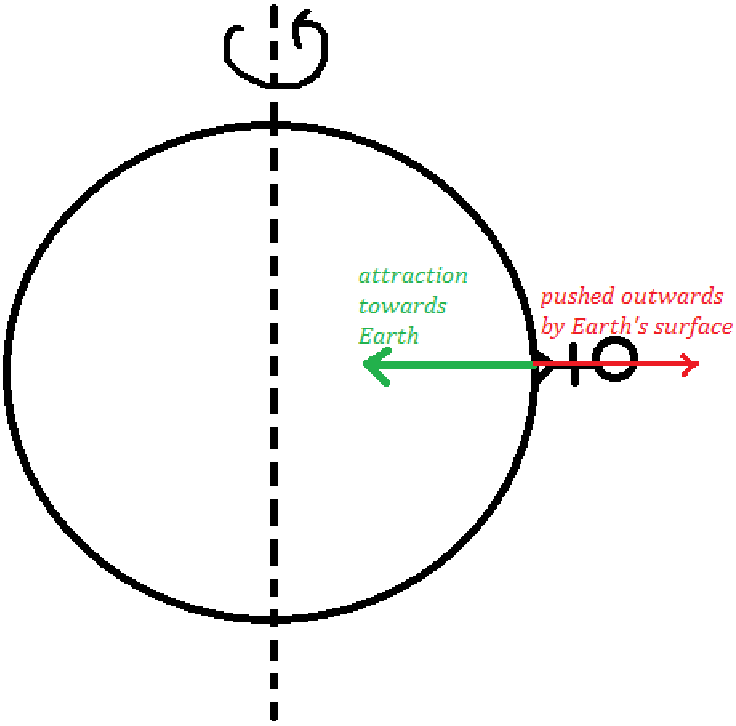
This question requires some knowledge beyond kinematics. I'll assume you haven't touch on forces yet. But I think it is good to have a bit of "physical intuition" – the feel on how it works. Let's just give it a try. I'll avoid calculations. Try to visualize what I'm saying.

Let's ask a question: why is this even a question? Given this question is asked, there must be a reason that someone will weigh differently if he runs at a different speed.

Common fact: the Earth rotates about an axis passing through the North and South poles, with a period of 24h (that is it spins one complete revolution or 2π radians in one day). Therefore, we as humans, standing on the Earth surface, experience a net centripetal acceleration pointing towards the centre of the Earth.

This chapter is before Forces, but hopefully it sounds intuitive enough. Let's just accept the fact that forces cause acceleration.

Your net acceleration is contributed by two forces: Earth attracting you inwards, and the surface of the Earth pushing you outwards (so that you do not get attracted into the abyss of darkness into the centre of the Earth). Each of these two forces give rise to a proportional acceleration.



Going back to the question, what does it mean to weigh? When we step on a weighing scale, we are pushing on the weighing scale, likewise the weighing scale is pushing back on you in equal strength (this is also known as Newton's 3rd Law of Motion). In other words, the weighing scale measures the outward force.

From what we've learnt, since you are undergoing circular motion, you have a net inward acceleration, therefore the attraction must be stronger than the outward push. If you are not rotating with the Earth, you have no net centripetal acceleration, and the attraction must equal to the outward push. This is exactly what the question wanted. In other words, to weigh the most, you must run at the same speed as the tangential speed of the equator of the Earth, in the opposite direction of its rotation. That is around 465 m/s.

(Yes the Earth rotates that fast. If the Earth were to suddenly stop spinning, you will be swung into the unknown of the outer space at a speed of 465m/s since Singapore is located around the equator. Just a hypothetical scenario to act as a prelude to Newton's Law of Motion.)   :)