Conservation of Momentum

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1 Introduction

In the previous chapter on forces, in particular the part on Newton's Third Law of Motion, there is a question on the scenario of two objects colliding. The result is that the total momentum in the system is conserved.

This simplifies a lot of problems, especially when the exact forces at every single moment of time is not known. To solve some problems, we do not need to know the exact process, but instead, we just need to know the initial and final states.

Disclaimer: I'll use a bit more Math in this set of notes. I'll try not to get out of hands. When I use vector notation, just realise that everything can be resolved into 3 coordinates and I am just writing one equation instead of three.

2 Momentum and Related Quantities

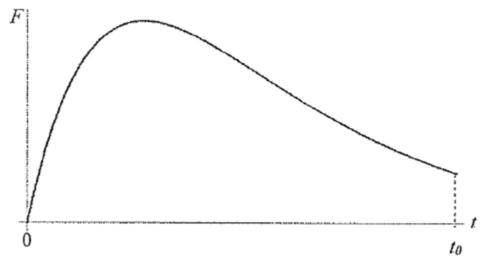
Taking us back to Newton's Law's of Motion, the net force $\vec{F}$ on an object is the rate of change in its momentum.

Just to recap, the momentum of an object with mass m and velocity vector $\vec{v}$ will have a momentum vector $\vec{p} = m\vec{v}$(i.e. the momentum is in the same direction as velocity). Its units is $kg\, m\, s^{-1}$.

If a system contains more than one object, just do a **vector addition/summation** of the momentum of all objects(Do look at the notes on vectors if you don't know what I am referring to).

The total change momentum of an object due to a force over a period of time is impulse, $\vec{J}$. It is calculated by $\vec{F}\Delta t$ or in integral form: $\int \vec{F}\, dt$.

C4. The twitch in an arm muscle develops a force F which can be measured as a function of time t, given by $F = F_0(\frac{t}{T})e^{-t/T}$, where $F - 0$ and T are constants. The function is plotted in the figure. If the ratio $\frac{t_0}{T} = 1$, what is the impulse between time $t = 0$ and $t = t_0$? (Hint: Make use of this integral. $\int_0^a xe^{-x}\, dx = 1 - (1 + a)e^{-a}$)(SJPO 2018)



Looks like SJPO do expect the usage of Calculus. Fortunately, we know the formula for impulse and they gave us the formula to work with.

Impulse is the change in momentum due to a force and is calculated by:

$$J = \int_0^{t_0} F\, dt = \int_0^{t_0} F_0(\frac{t}{T})e^{-t/T}\, dt$$

Since they gave us a formula, might as well use it. This technique is called integration by substitution(The trick is to treat it like a fraction even though Math teachers say not to).

$$\int_0^{t_0} F_0(\frac{t}{T})e^{-t/T}\, dt = \int_0^{t_0/T} F_0(\frac{t}{T})e^{-t/T}\frac{dt}{d\frac{t}{T}}\, d\frac{t}{T} = TF_0\int_0^{t_0/T} (\frac{t}{T})e^{-t/T}\, d\frac{t}{T}$$

As you can see, the equation we got is almost exactly like the equation they gave us. Since $\frac{t_0}{T} = 1$,

$$TF_0\int_0^{t_0/T} (\frac{t}{T})e^{-t/T}\, d\frac{t}{T} = TF_0(1 - (1 + 1)e^{-1}) = TF_0(1 - \frac{2}{e})$$

Just be careful when doing the Math. Be aware of which quantities are constants and which are variables.

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3 Internal and External Forces

You may realise that I use the word "system" a lot. For simplicity, just think of it as a set of objects. Just be mindful about which objects are in there.

When I have a collection of objects, the forces on any single object can be caused by either object in the system, or objects outside of the system. These are called internal and external forces respectively.

Fortunately for us, there is something we know about internal forces. Due to Newton's Third Law of Motion, if A exerts a force $\vec{F}$ on B, B exerts a force of $-\vec{F}$ on A(the negative sign is due to the opposite direction). If we generalize this for every single particle, the net internal force in a system will sum to zero.

Therefore, the rate of change in total momentum of a system is only due to external forces.

$$\vec{F}_{external} = \frac{d\vec{P}}{dt}$$

4 Centre of Mass

I have made the point that a system is typically made up of many objects. Even a ball is made up of atoms, however, we typically treat the ball as a point particle in the centre of the ball, instead of calculating the force on every individual atom.

We can do this because we assumed a point which all the mass concentrates at and the force acts on that point. That point is the centre of mass.

Here is a derivation(involving some calculus). The total momentum of a system is the sum of the momentum of individual particles.

$$\vec{P} = \vec{p}_1 + \vec{p}_1 + ... + \vec{p}_N = \sum_{n=1}^{N} \vec{p}_n$$

Then, the net force is

$$\vec{F} = \sum_{n=1}^{N} \frac{d\vec{p}_n}{dt} = \sum_{n=1}^{N} m_n \frac{d^2\vec{r}_n}{dt^2}$$

$\frac{d^2\vec{r}_n}{dt^2}$ means the second derivative of position vector with respect to time of the n^{th} particle, i.e. the acceleration. m_n is the mass of the n^{th} particle. The term on the most right just refers to the net force on the n^{th} particle.

I want to assume all the mass to concentrate at one point that fits Newton's Second Law, $F = Ma$. Then, the acceleration of that point $\frac{d^2\vec{R}}{dt^2}$ should be such that:

$$(m_1 + m_2 + ... + m_N)\frac{d^2\vec{R}}{dt^2} = \sum_{n=1}^{N} m_n \frac{d^2\vec{r}_n}{dt^2}$$

Where $M = m_1 + m_2 + ... + m_N$, a solution that fits will be:

$$\vec{R} = \frac{1}{M}\sum_{n=1}^{N} m_n \vec{r}_n$$

To be honest, I probably won't be able to understand this in Secondary 3 or 4 so I'll write it in a more understandable manner. The x,y,z coordinate of the centre of mass of a system of N particles each with coordinates x_n, y_n, z_n will be given by:

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + ... + m_Nx_N}{M}$$

$$y_{CM} = \frac{m_1y_1 + m_2y_2 + ... + m_Ny_N}{M}$$

$$z_{CM} = \frac{m_1z_1 + m_2z_2 + ... + m_Nz_N}{M}$$

If you observe the equations carefully, you will realise it is actually rather intuitive. You are sort of finding the average position of the masses.

C5. A thin wire is bent into the form of a three-sided shape as shown below. Each segment has equal length l. What is the height of the centre of mass from the bottom of the shape? (SJPO 2018)

If we investigate the formula for centre of mass, we can see that the centre of mass of a number of systems(each containing a number of particles) is also equal to when we regard each system as a point mass at its own centre of mass. (Try to digest what I mean)

It should be rather logical that the centre of mass of a straight uniform rod is in the middle of the rod. As we only want to find the height of the centre of mass, we just have to solve for one axis. Suppose each side has mass m,

$$y_{CM} = \frac{m\frac{l}{2} + m(0) + m\frac{l}{2}}{m + m + m} = \frac{l}{3}$$

You can try another method: by assuming that it is initially a closed box.

5 Principle of Conservation of Momentum

Using the previous argument regarding internal and external forces, it brings us to the principle of conservation of momentum.

If the vector sum of external forces is zero, the total momentum of a system is conserved,

This principle makes it convenient to solve problems regarding collision/explosions. Given no net external forces, we can just calculate the initial and final momentum and ensure they are equal.

Although momentum is conserved, kinetic energy may not(some energy may be lost as heat or sound) This allows us to classify the various types of collisions.

5.1 Elastic/Perfectly Elastic Collision

This is the ideal case. All kinetic energy is conserved. This also gives us a nice Mathematical result. Consider two objects with mass m_1, m_2 colliding along a straight line. All our velocities are positive in one direction and negative in the opposite. u represents velocities before collision and v represents velocity after collision.

Conservation of momentum gives

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$m_1(u_1 - v_1) = m_2(v_2 - u_2)$$

Conservation of kinetic energy gives

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2)$$

$$m_1(u_1 - v_1)(u_1 + v_1) = m_2(v_2 - u_2)(v_2 + u_2)$$

From both conservation of momentum and kinetic energy,

$$u_1 + v_1 = v_2 + u_2 \implies u_1 - u_2 = v_2 - v_1$$

$u_1 - u_2$ is the speed at which object 1 approaches object 2 before the collision. $v_2 - v_1$ is the speed at which object 2 separates from object 1 after collision. This is the principle that states that for an elastic collision, the relative speed of approach equals to the relative speed of separation.

5.2 Inelastic Collision

This is the case where not all kinetic energy is conserved. Nonetheless, momentum is still conserved. There is nothing much to say here except that there is one additional concept: the coefficient of restitution.

As we have shown previously, a perfectly elastic collision will have relative speed of approach equal to relative speed of separation. Perhaps, we should compare these two quantities: relative speed before and after collision.

$$e = \frac{\text{relative speed after collision}}{\text{relative speed before collision}}$$

In the case of a perfectly elastic collision, $e = 1$.

In the scenario, of a ball colliding with a stationary wall and rebounding, we will only be comparing the component of velocity normal/perpendicular to the wall.

5.3 Perfectly Inelastic Collision

This is a special case of inelastic collision, that is when the kinetic energy after the collision reaches a minimum, when coefficient of restitution is zero. This means after collision, the two objects are now sticking together and moving at the same velocity. Think of it as two plasticine ball that collided and deformed and got stuck together afterwards.

6 Centre of Momentum Frame

I probably should have brought in this concept earlier. A frame of reference refers to a coordinate system which an observer uses. The observer can be stationary on the ground or in a moving car.

Our Principle of Conservation of Momentum is based on Newton's Laws of Motion. It requires that an object not experiencing any force to not experience any spontaneous acceleration/change in momentum. Any such frame is known as an inertial frame.

In that case, any frame that moves at constant velocity with respect to an inertial frame is also an inertial frame. This is because all velocities are changed by the same value.

Out of all possible velocities, there is one particular velocity, where the total momentum in the system is zero. That is when the centre of mass remains stationary. In the case where the centre of mass is at the origin, this frame is known as the centre of mass frame.

Changing a frame of reference can sometimes simplify a problem.

C6. Let there be a 5kg and 3kg mass moving towards each other. Fill in the table.

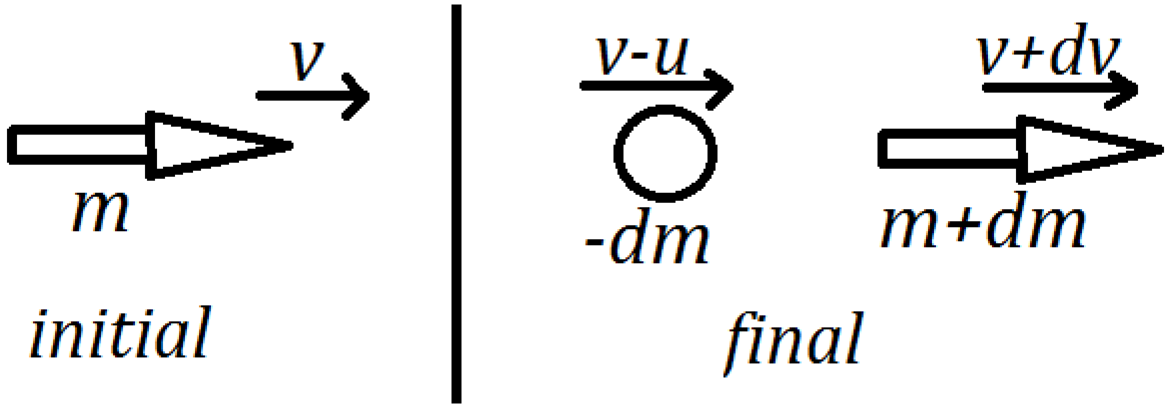
Initial speed/ m/s		Final speed/ m/s		Initial total KE/J	Final total KE/J	Type of Collision
5kg	3kg	5kg	3kg			
2	2				16	
2			2.5		10.6	
	20	5	5			

Now, let's do this in the centre of momentum/mass frame. What is the velocity of this frame with respect to the lab frame? Fill in the table again, for the same 3 cases, but for the COM frame.

Initial speed/ m/s		Final speed/ m/s		Initial total KE/J	Final total KE/J	Type of Collision
5kg	3kg	5kg	3kg			

C7. A rocket of mass M(that is initially at rest) ejects fuel at a constant rate at a speed of u relative to the rocket. What is the final speed of the rocket given that the final mass is αM? Ignore gravity.

This is a classic problem which I came across in JC(probably won't be able to solve it when in High School), but I thought it is interesting and there is something to be learnt from it.



Suppose at an instant, the rocket has mass m and is moving at velocity v rightwards. An instant later, its mass changes by an extremely small amount dm(a negative change), which means it ejected fuel of mass $-dm$. The fuel is ejected at u with respect to the rocket, which means the fuel has velocity $v-u$ rightwards. This ejection led to the velocity of the rocket to increase by dv.

Now that we have a picture of what is happening, time to bring in the Math. Using the Principle of Conservation of Momentum:

$$mv = -dm(v-u)+(m+dm)(v+dv) = -vdm+udm+mv+mdv+vdm+dmdv$$

$$0 = u\,dm + m\,dv + dm\,dv$$

When we write dv and dm, we are referring to a change that is so small, so close to zero. This makes $dm\,dv$ second-order(think of it as something really negligible) and can just be left out. After all:

$$-u\,dm = m\,dv$$

Time to integrate this and let the Math do the work. The mass of the rocket changed from M to αM and the velocity changed from 0 to the final velocity v_f which we are looking for. Note that u is a constant.

$$-u\int_M^{\alpha M}\frac{1}{m}dm = \int_0^{v_f} dv$$

$$v_f = -u\,ln\alpha$$

Does the negative sign make sense? Well, the mass definitely decreased, which makes $\alpha < 1$, which makes $ln\alpha$ negative. Therefore, with the negative sign, the final velocity is positive.

Don't worry if it is confusing. It will become more intuitive as time passes.