Conservation of Energy

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1 Introduction

In Physics, we study various quantities. If a quantity changes, we want to know how it changes. Then, there are certain quantities that do no change as time goes by. These quantities are said to be conserved. This can significantly simplify a lot of our problems.

This chapter mainly focuses on **energy**.

Before we start, let's keep one thing in mind. Everything in this chapter is strongly connected to Newton's Laws, $F = ma$. In fact, energy and forces are really two sides of the same coin. When an object slows down, we know there is a force acting on it, we know it is losing kinetic energy. Both explanations are different but equally correct for explaining the same phenomena.

2 Energy

Actually, what is energy? If we think about it, it doesn't really exist (at least in what we can see in our daily lives). I like to think of it as a quantity which scientists created, such that its value is conserved.

This portion is referenced from *Introduction to Classical Mechanics by David Morin*. I'll assume you haven't come in contact with the idea of work done and I'll start from Newton's Laws:

$$F = ma = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx}$$

This is chain rule in calculus. Doing some manipulation and integration:

$$\int_{x_1}^{x_2} F\,dx = \int_{v_1}^{v_2} mv\,dv = \frac{1}{2}m{v_2}^2 - \frac{1}{2}m{v_1}^2$$

Looking at the left-most term, it is the integral form of work done(the integral is more precise as the force may not be constant over the displacement), which may be familiar to most of us as $F\Delta s\ cos\theta$. Well, if the force and displacement are in the same direction, θ is zero and $cos\theta$ is just 1.

Moving the terms around:

$$\frac{1}{2}m{v_1}^2 + \int_{x_1}^{x_2} F\, dx = \frac{1}{2}m{v_2}^2$$

Let's see what each term means. If I were to exert a force on an object over a displacement, its velocity will change such that the above formula holds. The faster an object moves, $\frac{1}{2}mv^2$ also gets larger. This term is the kinetic energy(KE) of an object.

$$KE = \frac{1}{2}mv^2$$

So what is work done? Work done is what causes an energy transfer, causing energy in a system to change. Another way of thinking is:

$$E = \frac{1}{2}m{v_1}^2 + 0 = \frac{1}{2}m{v_2}^2 + \left(-\int_{x_1}^{x_2} F\, dx\right)$$

The work done is the energy lost by something else that causes energy to be gained by the object.

From the formula we can see that energy has units $Nm = J$(Joules).

2.1 Power

Power is just the rate of energy transfer/energy transfer per unit time with units $J/s = W$(Watts).

C1. Consider a block of mass m initially moving with velocity u on a rough horizontal floor with coefficient of friction μ. How far does it move before coming to a stop? Solve using (a) $F = ma$ (b) Energy.

Fortunately, the friction force is constant, μmg, and thus the acceleration is μg. Just plug into the kinematic equation and we get:

$$v^2 = u^2 + 2as \implies 0 = u^2 + 2(-\mu g)s \implies s = \frac{u^2}{2\mu g}$$

This is the $F = ma$ method, by considering the force and thus the motion of the object.

Next, using the energy approach. It started off with kinetic energy $\frac{1}{2}mu^2$ and ended with none. The energy loss is due to work done by friction.

$$W = -\frac{1}{2}mu^2 = (-\mu mg)(s) \implies s = \frac{u^2}{2\mu g}$$

Note that there can be negative work done if the force exerted is in the opposite direction of displacement.

The main purpose of this question is to show how force and energy are really equivalent in concept.

3 Conservation of Energy

We've come in contact with the concept of conservation of energy since primary school. This is the idea that the total energy in a system remains constant. If energy cannot be created or destroyed, it can only be converted from one form to another. In E1, the kinetic energy is probably lost due to heat generated. Kinetic energy is converted to thermal energy.

Now, what if there are no such dissipative forces, and I know how to calculate every form of energy the system has. Then, I can just make use of the conservation of energy to calculate the value for a certain energy.

4 Potential Energy

Often, people just consider the conversion between two categories of energy: kinetic and potential energy. Potential energy can be thought of as how much energy is available to increase an object's kinetic energy.

4.1 Gravitational Potential Energy(GPE)

Recall that I have made one assumption(or rather an approximation) that the gravitational acceleration is constant on Earth. This means an object on Earth experiences a gravitational force of mg.

Consider an object moving with an initial velocity upwards and I exert an upward force mg on the object, it will continue moving up with the same velocity as the vertical forces cancel out (ie no change in KE). Suppose I exerted the force over a height, h, then the work done by this force is mgh, which gives the **change** in gravitational potential energy.

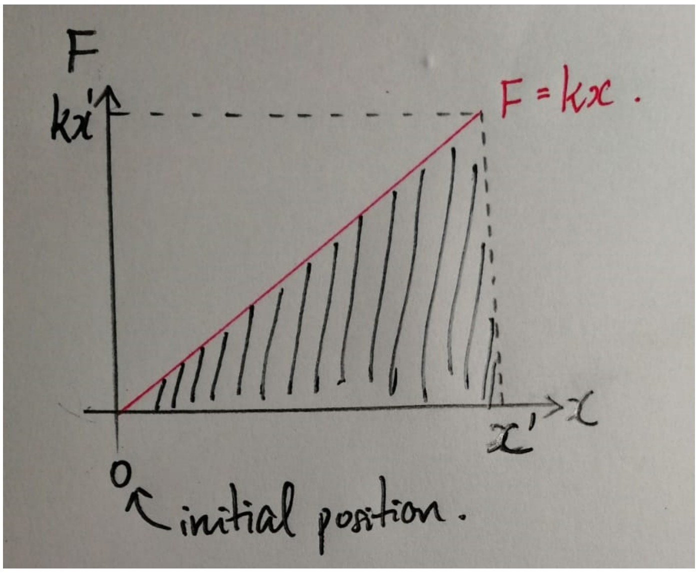
$$GPE_{final} = GPE_{initial} + mgh$$

4.2 Elastic Potential Energy(EPE)

A spring is slightly more complicated as the force varies with the length which the spring is extended. Fortunately, Mathematics is a powerful tool.

According to Hooke's Law(refer to chapter on forces), the force exerted by a spring is directly proportional to the extension of a spring. An external agent(maybe yourself) would need to exert exactly this force to keep on object in constant velocity(so kinetic energy remains constant).

As we remember, integrating gives the area below a graph. Work done is just the area below a F-x graph.



Therefore, the work done by the external agent to extend a spring by x', which is the energy input into the spring system is given by

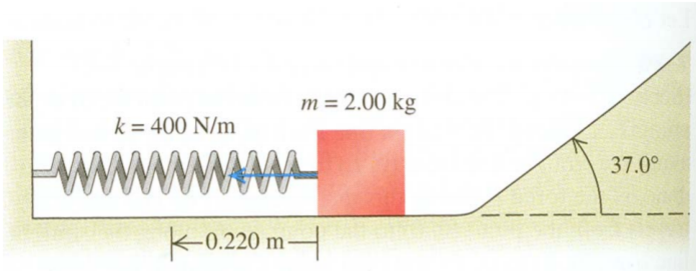
$$\int_0^{x'} F\,dx = \frac{1}{2}(kx')(x') = \frac{1}{2}kx'^2$$

Here we have it, a spring with spring constant k and extended by x will have elastic potential energy of $\frac{1}{2}kx^2$.

C2. A 2.00-kg block is pushed against a spring with negligible mass and force constant $k = 400\frac{N}{m}$, compressing it $0.220m$. When the block is released, it moves along a frictionless, horizontal surface and then up a frictionless incline with slope 37.0°.

(a) What is the speed of the block as it slides along the horizontal surface after having left the spring?

(b) How far does the block travel up the incline before starting to slide back down?



(a) When solving any problem, it is good to be aware of the **assumptions**(I shall bold them to bring your attention to them) that we are making. We shall consider our system as the spring and the block. Fortunately, **everything is frictionless**, so there is **no energy loss** from our system.

This is a very standard approach: due to the Principle of Conservation of Energy, the initial amount of energy equals to the final amount of energy. When it is still on the horizontal surface, GPE doesn't change, so let's just ignore it. The only energy that we have are EPE of the spring and KE of the object(well a real spring has kinetic energy too but in our case it is masslessm**massless spring**). Initially, the mass is at rest with zero velocity and kinetic energy; it leaves the spring after the spring returns to equilibrium(uncompressed) position.

$$EPE_{initial} + KE_{initial} = EPE_{final} + KE_{fianl}$$

$$\frac{1}{2}kx^2 + 0 = 0 + \frac{1}{2}mv^2 \Longrightarrow \frac{1}{2}\left(400\frac{N}{m}\right)(0.220^2) = \frac{1}{2}(2.00)v^2$$

$$v = 3.11m\,s^{-1}$$

(b) Now, the spring is completely out of the picture and the EPE term can be neglected; we just have to focus on KE and GPE. Fortunately, there is no friction anywhere again. When will it start to slide back? The block on the incline will start to slow down until a stop and then it will slide back down. So basically, we need to find when it stops instantaneously/when its KE is zero.

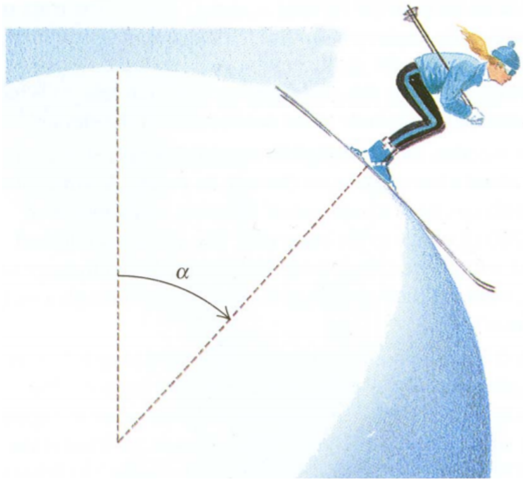
$$KE_{initial} + GPE_{initial} = KE_{final} + GPE_{final}$$

$$\frac{1}{2}mv^2 + 0 = 0 + mgh \Longrightarrow \frac{1}{2}(2.00)(3.11^2) = (2.00)(9.81)h$$

$$h = 0.493m$$

$$\text{distance along the incline} = 0.493 \div sin37^\circ = 0.820m$$

C3. A skier starts at the top of a very large frictionless snowball, with a very small initial speed, and skis straight down the side. At what point does she lose contact with the snowball and fly off at a tangent? That is, at the instant she loses contact with the snowball, what angle α does a radial line from the center of the snowball to the skier make with the vertical?



When I first saw this question in High School, my first question(and probably yours too), is what does it mean to lose contact. So long as the snowball is pushing against the skier, it means there is still contact. So it is probably right to say that there is contact as long as there is a normal force exerted by the snowball on the skier. Losing contact just refers to when this normal force reaches zero.

Fortunately, the snowball is **frictionless** and calculable **energy is conserved**. It is said that the initial speed is small, which is implying that the initial kinetic energy is negligible. As the skier slides down the snowball, gravitational energy is converted to kinetic energy. For now I'll define m as the mass of skier and R as the radius of the snowball and v as the speed of the skier at a certain angle. You will later realise that these variables are not necessary to get the answer but declaring certain useful variables is useful in visualizing the problem.

$$GPE_{initial} - GPE_{final} = KE \Longrightarrow mgR - mgR\,cos\alpha = mgR(1 - cos\alpha) = \frac{1}{2}mv^2$$

$$v^2 = 2gR(1 - cos\alpha)$$

How do we know the normal force, N? Since the skier is undergoing circular motion, while it remains in contact, the normal force and gravitational force along the radius must balance out to provide for the centripetal force.

$$m\frac{v^2}{R} = mg\;cos\alpha - N$$

When $N = 0$, that is when it loses contact,

$$v^2 = gR\,cos\alpha$$

Solving the equations simultaneously,

$$2gR(1 - cos\alpha) = gR\,cos\alpha$$

$$cos\alpha = \frac{2}{3}$$

This value of α is when it will lose contact.

The above two questions are taken from University Physics, Young and Freedman/MIT Opencourseware.

5 Conservative and Non-conservative Forces

Consider a ball being thrown upwards and then falling back down. KE is converted to GPE and then back to KE again. If the system allows for such a transfer of energy between kinetic and potential energy, the force related to the potential energy is said to be conservative.

Friction on the other hand is non-conservative. As KE is lost due to friction, the energy lost cannot be converted back to KE.

One characteristic of a conservative force is that the force and corresponding potential energy is **only dependent on position**. What this means is that regardless of how I move the object from A to B, the energy change is the same.

In this case, let's consider a system with potential energy $V(x)$ that is only dependent on position. An increase in potential energy will correspond to negative work done on the object.

$$V(x_2) - V(x_1) = -F \cdot (x_2 - x_1)$$

$$\Delta V = -F\Delta x \Longrightarrow -\frac{\Delta V}{\Delta x} = F$$

If I were to do it the calculus way,

$$-\frac{dV}{dx} = F \; or \; V = -\int F\,dx$$

Fortunately, a lot of forces we deal with(gravity, spring force etc) are conservative, which makes it easy to find a formula for the corresponding energy.